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# ON A MONOTONICITY PROPERTY OF MEASURES OF DIRECTED-DIVERGENCE

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For any three given probability distributions $P$, $Q$, $R$, it is shown that Csiszer's measure of directed divergence of $(kP + R)/(k + 1)$ from $(kQ + R)/(k + 1)$ is a monotonic increasing function of $k$.

## 1. INTRODUCTION

In Euclidean geometry, let $P'$, $Q'$ be points which divide the sides $RP$, $RQ$ in the same ratio $k : 1$, then it is known that the length of $P' Q'$ is $k/(k + 1)$ times the length of $PQ$ and $P' Q'$ is parallel to $PQ$.

Now let

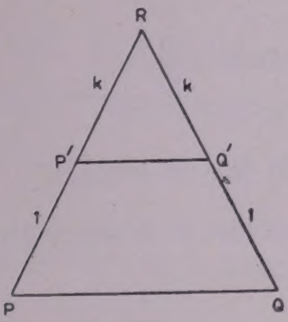


FIG. 1.

$$P = (p_1, p_2, \ldots, p_n),\ Q = (q_1, q_2, \ldots, q_n),\ R = (r_1, r_2, \ldots, r_n) \qquad \ldots(1)$$

be three probability distributions and let

$$P' = \frac{kp_1 + r_1}{k + 1},\ \frac{kp_2 + r_2}{k + 1}, \ldots, \frac{kp_n + r_n}{k + 1}$$

$$Q' = \frac{kq_1 + r_1}{k + 1},\ \frac{kq_2 + r_2}{k + 1}, \ldots, \frac{kq_n + r_n}{k + 1}. \qquad \ldots(2)$$

We now find directed divergences $D(P' : Q')$ and $D(P : Q)$, and motivated by the above result in Euclidean Geometry, ask the following questions :

(a) Is $D(P' : Q')/D(P : Q) = k/(k+1)$ ?

(b) Is $D(P' : Q') \leqq D(P : Q)$?

(c) Is $D(P' : Q')$ monotonic increasing function $k$ ?

Now there are a large number of measures of directed divergences[4]. The most important of these is the measure due to Csiszer[1], viz.

$$D(P: Q) = \sum_{i=1}^{n} q_i f\left( \frac{p_i}{p_i} \right) \qquad \ldots(3)$$

where $f(\,.\,)$ is a convex twice-differentiable function for which $f(1) = 0$. It can easily be shown that

$$D(P : Q) \geqslant 0 \qquad \ldots(4)$$

$$D(P : Q) = 0 \text{ iff } P = Q \qquad \ldots(5)$$

and

$D(P : Q)$ is a convex function of $p_1, p_2, \ldots, p_n$ as well as of $q_1, q_2, \ldots, q_n$.

The measure (3) includes the following measures of directed divergence as special cases:

(i) Kullback-Leibler[5] measure of directed divergence

$$D_1(P : Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}. \qquad \ldots(6)$$

This is obtained by putting $f(x) = x \ln x$ in (3).

(ii) Havrda-Charvat[2] measure of directed divergence

$$D_2(P : Q) = \frac{1}{(\alpha - 1)}\left( \sum_{i=1} p^{\alpha}_i \; q_i^{1-\alpha} - 1 \right), \alpha > 0, \alpha \neq 1. \qquad \ldots(7)$$

This is obtained by putting $f(x) = (x^{\alpha} - x)/(\alpha - 1)$ in (3).

(iii) Sharma-Taneja[7] measure of directed divergence

$$D_3(P : Q) = \frac{1}{(\alpha - \beta)}\left( \sum_{i=1}^{n} p_i^{\alpha} \, q_i^{1-\alpha} - \sum_{i=1}^{n} p_i^{\beta} \, q_i^{1-\beta} \right) \qquad \ldots(8)$$

where

$$\alpha > 1,\ 0 < \beta < 1 \text{ or } 0 < \alpha < 1 \text{ and } \beta > 1. \qquad \ldots(9)$$

This is obtained by putting $f(x) = \dfrac{(x^\alpha - x^\beta)}{(\alpha - \beta)}$ in (3). ...(10)

(iv) Kapur's[4] measure of directed divergence is

$$D_4(P:Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} - \frac{b}{a^c} \sum_{i=1}^{n} (q_i + ap_i) \ln \frac{1 + a\dfrac{p_i}{q_i}}{1+a}$$

where $a^{c-1} > b$. ...(11)

This is obtained by putting

$$f(x) = x \ln x - \frac{b}{a^c}(1 + ax) \ln\left(\frac{1+ax}{1+a}\right) \quad \text{...(12) in (3).}$$

It is obvious that any theorem proved for Csiszer's measure (3) will continue to hold for the measures (6), (7), (8) and (11).

In the present paper, we shall show that for Csiszer's measure the answer to question (a) is in the negative, while the answers to questions (b) and (c) are in the affirmative. The answer to question (a) is also in the affirmative if equality sign there is replaced by sign $\leqslant$.

All the properties are highly desirable and are expected from every measure which seeks to measure discrepency or distance in some sense.

## 2. A Basic Theorem and Its Consequences

*Theorem*—Csizser's measure of directed divergence

$$D(P' : Q') = D\left(\frac{kP + R}{k+1} : \frac{kQ + R}{k+1}\right) \quad \text{...(13)}$$

is a monotonic increasing function of $k$, ($k \geqslant 0$) and increases from 0 to $D(P:Q)$ as $k$ increases from 0 to $\infty$.

Proof : Let

$$g(k) = D\left(\frac{kP + R}{k+1} : \frac{kQ + R}{k+1}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{kq_i + r_i}{k+1}\right) f\left(\frac{kp_i + r_i}{kq_i + r_i}\right) \quad \text{...(14)}$$

so that

$$g'(k) = \frac{1}{(1+k)^2}\left[\sum_{i=1}^{n} (q_i - r_i) f\left(\frac{r_i + kp_i}{r_i + kq_i}\right)\right.$$

(*equation continued on p.* 854)

$$+ \sum_{i=1}^{n} \frac{(1+k) r_i (p_i - q_i)}{(r_i + kq_i)} f' \left( \frac{r_i + kp_i}{r_i + kq_i} \right) \Big] \quad ...(15)$$

$$= \frac{R\ (k)}{(1+k)^2} \text{ (say).} \quad ...(16)$$

Now

$$R'(k) = \sum_{i=1}^{n} \frac{(1+k)\ r_i^2\ (p_i - q_i)^2}{(r_i + kq_i)^3} f'' \left( \frac{r_i + kp_i}{r_i + kq_i} \right) \quad ...(17)$$

since $f(.)$ is convex, $f''(.) \geqslant 0$ and $R'(k)$ vanishes when $p_i = q_i$ for each $i$ so that $R'(k) \geqslant 0$. Also from (15) and (16), $R(0) = (0)$, since $f(1) = 0$ and $\sum_{i=1}^{n} (p_i - q_i) = 0$ so. Thus $R(k) \geqslant 0$. Thus $g'(k) \geqslant 0$ so that $g(k)$ is a monotonic increasing function of $k$. Also $g(0) = 0$ and $g(\infty) = D(P:Q)$. Thus the theorem is proved.

We deduce the following corollaries from this theorem.

*Corollary* 1—The answers to questions (b) and (c) of section 1 are in the affirmative.

*Corollary* 2—For each of the measures due to Kullback-Leibler[5], Havrda-Charvat[2], Sharma-Taneja and Kapur[4], the directed divergence of $\frac{(kP + R)}{(k+1)}$ from $\frac{(kQ + R)}{(k+1)}$ is a monotonic increasing function of $k$ which increases from 0 to the the value of directed divergence of $P$ from $Q$, as $k$ increases from 0 to $\infty$.

*Corollary* 3—For Renyi's measure of directed divergence defined by

$$D_5(P:Q) = \frac{1}{(\alpha - 1)} \ln \sum_{i=1}^{n} p_i^{\alpha}\ q_i^{1-\alpha},\ \alpha \neq 1,\ \alpha > 0 \quad ...(18)$$

$D_5 \left[ \frac{(kP + R)}{(k+1)} : \frac{(kQ + R)}{(k+1)} \right]$ is a monotonic increasing function of $k$ which increases from 0 to $D_5(P:Q)$ as $k$ increases from 0 to $\infty$.

PROOF: From (7) and (18)

$$D_5(P:Q) = \frac{1}{(\alpha - 1)} \ln [(\alpha - 1) D_2(P:Q) + 1] \quad ...(19)$$

so that

$$D_5(P':Q') = \frac{1}{[(\alpha - 1) D_2(P':Q') + 1]} \frac{d}{dk} D_2(P':Q')$$

(*equation continued on p.* 855)

$$= \frac{1}{\sum_{i=1}^{n} p_i'^{\alpha} \; q_i'^{1-\alpha}} \; \frac{d}{dk} \; D_2 (P' : Q'). \qquad \ldots(20)$$

Since $\sum_{i=1}^{n} p_i^{\alpha} \quad q_i^{1-\alpha} > 0$ and $D_2$ $(P' : Q')$ is a monotonic increasing function of $k$ it follows from (20) that $D\,(P' : Q')$ is also a monotonic increasing function of $k$.

*Corollary* 4—For Kapur's[3] measare of directed divergence defined by

$$D_6\,(P : Q) = \frac{1}{(\alpha - \beta)} \ln \frac{\sum_{i=1}^{n} p_i^{\alpha} \quad q_i^{1-\alpha}}{\sum_{i=1}^{n} p_i^{\beta} \, q_i^{1-\beta}} \quad \begin{array}{l} 0 < \alpha < 1,\, \beta > 1, \text{ or} \\ ; \\ 0 < 1 < i,\, \alpha < 1. \end{array} \qquad \ldots(21)$$

$D_6\,(P' : Q')$ is a monotonic increasing function of $k$ which increases from 0 to $D_6$ $(P : Q)$ as $k$ increases from 0 to $\infty$.

PROOF :

$$D\,(P' : Q') = \frac{\alpha - 1}{\alpha - \beta} \left[ \frac{1}{\alpha - 1} \ln \sum_{i=1}^{n} p_i'^{\alpha} \cdot q_i'^{1-\alpha} \right]$$

$$+ \frac{1 - \beta}{\alpha - \beta} \left[ \frac{1}{\beta - 1} \; \ln \sum_{i=1}^{n} p_i^{\beta} \; q_i^{1-\beta} \right]. \qquad \ldots(22)$$

From Corollary 3, the expressions within the square brackets are monotonic increasing functions of $k$. Also if the conditions on $\alpha$, $\beta$ given in (21) are satisfied, then the coefficients of both the expressions in square brackets are positive. It follows that $D_6\,(P' : Q')$ is a monotonic increasing function of $k$.

*Corollary* 5—: The answers to questions (b) and (c) of section 1 are in the affirmative for both Renyi's[6] and Kapur'[3] second measure of directed divergence.

## 3. DISCUSSION OF QUESTION (A) OF SECTION 1

Since each of the measures of directed divergence due to Csiszer[1], Kullback-Leibler[5], Havrda-Charvat[2], Sharma-Taneja[7] and Kapur[4] is a convex function of both $P$ and $Q$, it follows that for each of these

$$D[mP + (1 - m) R : mQ + (1 - m) R]$$
$$\leqslant mD(P : Q) + (1 - m) D(R : R) = mD(P : Q), 0 \leqslant m \leqslant 1. \quad ...(23)$$

Putting $m = \frac{k}{k+1}$, we get

$$D\left( \frac{kP + R}{k+1} : \frac{kQ + R}{k+1} \right) \leqslant \frac{k}{(k+1)} D(P : Q) \quad ..(24)$$

or

$$D(P' : Q') \leqslant \frac{k}{(1+k)} D(P : Q). \quad ...(25)$$

Thus the answer to question (a) for all these measures is in the affirmative if the equality sign there is replaced by $\leqslant$ sign.

By taking special probability distributions, it can easily be shown that in (25) equality sign does not always hold for Csiszer's measure or for any of its special cases.

The equality sign will however hold for the metrics

$$D(P : Q) = [\sum_{i=1}^{n} (p_i - q_i)^{2r})^{1/2r}, \quad r = 1, 2, 3 \quad ...(26)$$

or

$$D(P : Q) = [\sum_{i=1}^{n} | p_i - q_i |^r]^{1/r}, \quad r = 1, 2, 3, ... \quad ...(27)$$

It may be noted that while these metrics are symmetric and satisfy the triangle inequality, our directed divergence measures do not satisfy these properties.

If we take symmetric measures of directed divergence defined by

$$J(P : Q) = D(P : Q) + D(Q : P) \quad ...(28)$$

questions (b) and (c) are still answered in the affirmative, while question (a) is answered in the affirmative if we replace equality sign by inequality sign.

## 4. An Interesting Property of Symmetric Diverwench

When $D(P : Q)$ is a convex function of both $P$ and $Q$, we get

$$D\left( \frac{kP + Q}{k+1} : R \right) \leqslant \frac{k}{(k+1)} D(P : R) + \frac{1}{(k+1)} D(Q : R) \quad ...(29)$$

$$D\left( R : \frac{kP + Q}{k+1} \right) \leqslant \frac{k}{(k+1)} D(R : P) + \frac{1}{(k+1)} D(R : Q) \quad ...(30)$$

so that

$$J\left(\frac{kP+Q}{k+1}:R\right) \leqslant \frac{k}{(k+1)} J(P:R) + \frac{1}{(k+1)} J(Q:R). \quad \ldots(31)$$

Similaraly

$$J\left(\frac{kQ+R}{k+1}:P\right) \leqslant \frac{k}{(k+1)} J(Q:P) + \frac{1}{(k+1)} J(R:P) \quad \ldots(32)$$

and

$$J\left(\frac{kR+P}{k+1}:Q\right) \leqslant \frac{k}{(k+1)} J(R:Q) + \frac{1}{(k+1)} J(P:Q). \quad \ldots(33)$$

From (31) – (33) we get

$$J\left(\frac{kP+Q}{k+1}:R\right) + J\left(\frac{kQ+R}{k+1}:P\right) + J\left(\frac{kR+P}{k+1}:Q\right)$$

$$\leqslant J(P:Q) + J(Q:R) + J(R:P). \quad \ldots(34)$$

In particular if $k = 1$,

$$J\left(\frac{P+Q}{2}:R\right) + J\left(\frac{Q+R}{2}:P\right) + (J\left(\frac{R+R}{2}:Q\right)$$

$$\leqslant J(P:Q) + J(Q:R) + J(R:P). \quad \ldots(35)$$

The inequalities (34) and (35) will hold for both Euclidean metrics as well as for Csiszer's symmetric divergence and its special cases.

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# BAYES APPROACH TO PREDICTION IN SAMPLES FROM GAMMA POPULATION WHEN OUTLIERS ARE PRESENT

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This paper deals with the problem of predicting order statistics in samples from a gamma population when an outlier is present. First, predictive distribution of the $r$th order statistics is obtained where an outlier of type $\theta\delta$ is present. As a special case with $r = 1$ and $r = n$, predictive distributions of minimum and maximum are obtained. In this connection, some identities are derived. Next outlier of type $\theta + \delta$ is dealt with. In this case, predictive distribution of the $r$th order statistics is expressed in terms of hypergeometric functions. Small tables give the variances of minimum and maximum.

## 1. INTRODUCTION

In this paper, few concepts like order statistics, prediction and outliers are dealt with. Outlier problem has received much attention recently. Kitagawa[5] uses Bayes approach to analyse when outliers are present. Barnett and Lewis[2] is a text devoted entirely to outliers. Lingappaiah[7,8] deals with the estimation problem when outliers are present and expresses estimates in terms of hypergeometric functions. Regarding prediction, Lingappaiah[6], predicts an order statistics using classical approach while, Dunsmore[3] and Lingappaiah[9,11,12] use Bayes approach to the same problem of prediction. In this paper, a different situation is taken up. That is, in the samples now, outliers are present. First, predictive distribution of the $r$th order statistics a sample from a gamma population is obtained when an outlier of type $\theta\ \delta$ is present. As special cases, by setting $r = 1$ and $r = n$ predictive distributions of the minimum and maximum are obtained. In this connection, some identities are derived. Next, outlier of type $\theta + \delta$ is considered and in this case, predictive distribution of the $r$th order statistics is expressed in terms of hypergeometric functions given in Erdelyi *et al.*[4]. Variances of minimum and maximum are given for small values of $\delta$, $\alpha$ and $n$.

## 2. PREDICTIVE DISTRIBUTION (OUTLIER OF TYPE $\theta\delta$)

If $x$ has the gamma distribution

$$f(x) = e^{-\theta x}\,(\theta x)^{\alpha-1}\,\theta/\Gamma(\alpha),\ x > 0,\ \alpha > 0 \qquad \text{...(1)}$$

then (with $\alpha$ as an integer), one has

$$F(x) = 1 - \sum_{k=0}^{\alpha-1} e^{-\theta x} (\theta x)^k / k!. \quad \text{...(1a)}$$

Distribution of the $r$th order statistics, in a sample of size $n$ (Balakrishnan[1] when an outlier is present is

$$h(x) = (a)\left[(r-1) F^{r-2} (1-F)^{n-r} G(x) f(x) + F^{r-1} (1-F)^{n-r} g(x) + (n-r) F^{r-1} (1-F)^{n-r-1} [1 - G(x)] f(x)\right]. \quad \text{...(2)}$$

Where $a = \binom{n-1}{r-1}$ and $f(x) = f$ and $F(x) = F$ are the density and distribution function of all those $x$'s which are not outliers while $g(x)$ and $G(x)$ are those of an outlier. If the outlier is of type $\theta\delta$, then from (1), we have

$$g(x) = e^{-\theta\delta x} (\theta\delta x)^{\alpha-1} \theta\delta / \Gamma(\alpha) \quad \text{...(3)}$$

and if the $r$th order statistics is $x_{(r)} = u$, then we have (2) as,

$$\begin{aligned} f(u \mid \theta) = [a/\Gamma(\alpha)] [(r-1) \{1 - z(\theta u)\}^{r-2} \{z(\theta u)\}^{n-r} \{1 - z(\theta\delta u)\} \\ \times \{e^{-\theta u} (\theta u)^{\alpha-1} \theta\} + (1 - z(\theta u)\}^{r-1} \{z(\theta u)\}^{n-r} \{e^{-\theta\delta u} \\ \times (\theta\delta u)^{\alpha-1} \theta\delta\} + (n-r) \{1 - z(\theta u)\}^{r-1} \{z(\theta u)\}^{n-r-1} \\ \{z(\theta\delta u)\} \{e^{-\theta u} (\theta u)^{\alpha-1} \theta\}] \end{aligned} \quad \text{...(4)}$$

where

$$z(\theta u) = \sum_{k=0}^{\alpha-1} e^{-\theta u} (\theta u)^k / k!. \quad \text{...(4a)}$$

Now (4) is

$$\begin{aligned} f(u/\theta) = [a/\Gamma(\alpha)] [(r-1) (\Omega_0) \{z(\theta u)\}^m \{1 - z(\theta\delta u)\} \{e^{-\theta u} (\theta u)^{\alpha-1} \theta\} \\ + (\Omega) \{z(\theta u)\}^m \{e^{-\theta\delta u} (\theta\delta u)^{\alpha-1} (\theta\delta)\} \\ + (\Omega) (n-r) \{z(\theta u)\}^{m-1} \{z(\theta\delta u)\} \{e^{-\theta u} (\theta u)^{\alpha-1} \theta\}] \end{aligned} \quad \text{...(5)}$$

where

$$m = n - r + j, \; (\Omega_0) = \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^j, \; (\Omega) = \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j.$$

Now,

$$\{z(u)\}^m = e^{-\theta m u} \sum_{t=0}^{(\alpha-1)m} A_t (\alpha, m (\theta u)^t \quad \text{..(6)}$$

where $A_t(\alpha, m)$ is the coefficient of $x^t$ in the expansion of

$(\sum_{x=0}^{\alpha-1} x^x/k!)^m$ and satisfies the equation

$$A_t(\alpha, m) = A_t(\alpha, m-1) + A_{t-1}(\alpha, m-1) + \ldots + \frac{1}{(\alpha-1)!} A_{t-\alpha+1} \times (\alpha, m-1). \quad \ldots(7)$$

Using (6), we get (5) as

$$f(u \mid \theta) = [a/\Gamma(\alpha)]\, [(r-1)(\Omega_0) \sum_{t=0}^{(\alpha-1)-m} A_t(\alpha, m) \quad \ldots(8)$$

$$\{\phi(m+1)(\theta u)^{t+\alpha-1}\theta - \sum_{k=0}^{\alpha-1} (\delta^k/k!)\, \phi(m+\delta+1)(\theta u)^{t+\alpha+k-1}.\, \theta\}$$

$$+ (\Omega) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\, \delta^\alpha\, \phi(m+\delta)(\theta u)^{t+\alpha-1}.\, \theta$$

$$+ (n-r)(\Omega) \sum_{t=0}^{(\alpha-1)(m1)} A_t(\alpha, m-1)\, \phi(m+\delta) \sum_{k=0}^{\alpha-1} (\delta^k/k!) \times (\theta u)^{t+\alpha+k-1}.\, \theta]$$

where

$$\phi(m) = e^{-\theta m u}.$$

Now, it is known that the estimate of $\theta$ is

$$\hat{\theta} = \sum_{i=1}^{r} x_{(i)} + (n-r)\, x_{(r)} \quad \ldots(9)$$

where $x_{(i)}$ is the $i$th order statistics.

Then $\hat{\theta}$ has the gamma distribution

$$f(\hat{\theta} \mid \theta) = e^{-\theta\hat{\theta}}(\theta\hat{\theta})^{r-1}\theta/\Gamma(r). \quad \ldots(10)$$

If the prior for $\theta$ is

$$g(\theta) = e^{-\theta h}(\theta h)^{g-1}/\Gamma(g)(1/h) \quad \ldots(11)$$

then we have

$$f(\theta \mid \hat{\theta}) = e^{-\theta H}(\theta H)^{G-1}\theta/\Gamma(G) \quad \ldots(12)$$

where

$$H = h + \hat{\theta},\ G = g + r.$$

From (8) and (12), one has

$$f(u \mid \theta) f(\theta \mid \hat{\theta}) = [aH^G/\Gamma(\alpha)\,\Gamma(G)]\,[\,(r-1)(\Omega_0) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m) \qquad \text{...(13)}$$

$$\{\phi_0(m+1)(\theta u)^{t+\alpha-1}.\,\theta^G - \sum_{k=0}^{\alpha-1} (\delta^k/k!)\,\phi_0(m+\delta+1)(\theta u)^{t+k+\alpha-1}.\,\theta^G\}$$

$$+ (\Omega) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\,\phi_0(m+\delta)\,\delta^\alpha\,(\theta u)^{t+\alpha-1}.\theta^G$$

$$+ (n-r)(\Omega) \sum_{t=0}^{(\alpha-1)(m-1))} A_t(\alpha, m-1) \sum_{k=0}^{\alpha-1} (\delta^k/k!)\,\phi_0(m+\delta)$$

$$\times (\theta u)^{t+k+\alpha-1}.\,\theta^G]$$

where

$$\phi_0(m) = e^{-\theta[H+mu]}. \qquad \text{...(13a)}$$

From (13), we get the predictive distribution of the $r$th order statistics, $x_{(r)} = u$ as

$$f(u \mid \hat{\theta}) = \int_0^\infty f(u \mid \theta) f(\theta \mid \hat{\theta})\, d\theta$$

$$= \left[\frac{aH^G}{\Gamma(\alpha)\,\Gamma(G)}\right]\left[(r-1)(\Omega_0) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\right.$$

$$\times \left\{\frac{\Gamma(t+\alpha+G)^{t+\alpha-1}}{[H+(m+1)u]^{t+\alpha+G}} - \sum_{k=0}^{\alpha-1} \left(\frac{\delta^k}{k!}\right)\right.$$

$$\times \left.\frac{\Gamma(t+k+\alpha+G)\,u^{t+k+\alpha-1}}{[H+(m+\delta+1)u]^{t+k+\alpha+G}}\right\} + (\Omega)\left\{\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\right.$$

$$\times \left.\frac{\delta^\alpha\,\Gamma(t+\alpha+G)u^{t+\alpha-1}}{[H+(m+\delta)u]^{t+\alpha+G}}\right\} + (n-r)(\Omega) \sum_{t=0}^{(\alpha-1)(m-1)} A_t(\alpha, m-1)$$

$$\left.\sum_{k=0}^{\alpha-1} \left(\frac{\delta^k}{k!}\right) \frac{\Gamma(t+k+\alpha+G)\,u^{t+k+\alpha-1}}{[H+(m+\delta)u]^{t+k+\alpha+G}}\right]. \qquad \text{...(14)}$$

Using (14), one can obtain the probability

$p(u \geqslant u_0) = \beta$ for set $\beta$.

Actually, if $\alpha = 1$, we get from (14)

$$p\,(u \geqslant u_0) = a\left[(r-1)\,(\Omega_0)\left\{\frac{1}{(m+1)}\left(\frac{H}{H+(m+1)u_0}\right)^G - \frac{1}{(m+\delta+1)}\left(\frac{H}{H+(m+\delta+1)u_0}\right)^G\right\} + (\Omega)\left(\frac{n-r+\delta}{m+\delta}\right)\left(\frac{H}{H+(m+\delta)u_0}\right)^G\right]. \quad \text{...(14a)}$$

*Illustrative Example*

Following is a simulated sample of size 5 from a gamma population (with $\alpha = 1$), where $\theta = 1/1000$ in (1).

111, 253, 276, 471, 548 ...(14b)

If $r = 2$ in (9), then from (14b), we have $\hat{\theta} = 1123$.

Suppose $g = h = 2$, in (11), then $G = 4$, $H = 1125$. Table I gives $p\,(u \geqslant 100)$ for various values of $\delta$ using (14).

TABLE I

| $\delta$ | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p\,(u \geqslant 100)$ $(n = 2,\ r = 1)$ | 0.6061 | 0.5197 | 0.3855 | 0.2962 |
| $p\,(u \geqslant 100)$ $(n = 3,\ r = 2)$ | 0.8356 | 0.7820 | 0.7042 | 0.6526 |

From (14), we get

$$\int_0^\infty f(u \mid \hat{\theta})\,du = \left[\frac{a}{\Gamma(\alpha)}\right]\left[(r-1)\,(\Omega_0)\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\left\{\frac{\Gamma(t+\alpha)}{(m+1)^{t+\alpha}} - \sum_{k=0}^{\alpha-1}\left(\frac{\delta^k}{k!}\right)\frac{\Gamma(t+k+\alpha)}{(m+\delta+1)^{t+k+\alpha}}\right\}\right.$$

$$+ (\Omega)\sum_{t=0}^{(\alpha-1)m}\left\{\frac{\delta^\alpha\,\Gamma(t+\alpha)}{(m+\delta)^{t+\alpha}}\right\}(A_t(\alpha, m))$$

(*equation continued on p.* 863)

$$+ (n - r)(\Omega) \sum_{t=0}^{(\alpha-1)(m-1)} A_t(\alpha, m-1) \sum_{k=0}^{\alpha-1} \left(\frac{\delta^k}{k!}\right)$$

$$\times \left.\frac{\Gamma(t+k+\alpha)}{(m+\delta)^{t+k+\alpha}}\right]. \qquad \ldots(15)$$

Also from (14), one has

$$E(u^s) = \frac{aH^s\,\Gamma(G-s)}{\Gamma(\alpha)\,\Gamma(G)}\Bigg[(r-1)(\Omega_0) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)$$

$$\left\{\frac{\Gamma(t+s+\alpha)}{(m+1)^{t+s+\alpha}} - \sum_{t=0}^{\alpha-1}\left(\frac{\delta^k}{k!}\right)\frac{\Gamma(t+s+\alpha)}{(m+\delta+1)^{t+k+\alpha+s}}\right\}$$

$$+ (\Omega)\left\{\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\ \frac{\Gamma(t+s+\alpha)\,\delta^\alpha}{(m+\delta)^{t+s+\alpha}}\right.$$

$$+ (n-r) \sum_{t=0}^{(\alpha-1)(m-1)} A_t \sum_{k=0}^{(\alpha-1)}\left(\frac{\delta^k}{k!}\right)(\alpha, m-1)$$

$$\left.\left.\frac{\Gamma(t+k+s+\alpha)}{(m-\delta)^{t+k-s+\alpha}}\right\}\right]. \qquad \ldots(15a)$$

(2a) *Minimum* $(r = 1)$, $x_{(1)} = v$

If $r = 1$, then we have from (14), the predictive distribution of the minimum, $x_{(1)} = v$, [now, $m = n - 1$, and first term vanishes], as

$$f(v \mid \hat{\theta}) = \left[\frac{H^G}{\Gamma(\alpha)\,\Gamma(G)}\right]\Bigg[\sum_{t=0}^{(\alpha-1)(n-1)} A_t(\alpha, n-1)$$

$$\times \frac{\delta^\alpha\,\Gamma(t+\alpha+G)\,v^{t+\alpha-1}}{[H+(n+\delta-1)v]^{t+\alpha+G}} + (n-1)\sum_{t=0}^{(\alpha-1)(n-2)}$$

$$\times A_t(\alpha, n-2)\sum_{k=0}^{\alpha-1}\left(\frac{\delta^k}{k!}\right)\frac{\Gamma(t+\alpha+k+G)\,v^{t+k+\alpha-1}}{[H+(n+\delta-1)v]^{t+k+\alpha+G}}\Bigg].$$

$$\ldots(16)$$

From (16), we have an identity using

$$\int_0^\infty f(v \mid \hat{\theta})\, d\theta = 1 \text{ as,}$$

$$\sum_{t=0}^{(\alpha-1)(n-1)} A_t(\alpha, n-1) \frac{\delta^\alpha \Gamma(t+\alpha)}{\Gamma(\alpha)(n+\delta-1)^{t+\alpha}}$$

$$+ \sum_{t=0}^{(\alpha-1)(n-2)} A_t(\alpha, n-2) \sum_{k=0}^{\alpha-1} \left(\frac{\delta^k}{k!}\right) \frac{\Gamma(t+k+\alpha)}{\Gamma(\alpha)(n+\delta-1)^{t+k+\alpha}}$$

$$= 1. \qquad \ldots(16a)$$

If $r = 1$, then (15a) (or from (16)) gives,

$$E(v^s) = \frac{H^s \Gamma(G-s)}{\Gamma(\alpha)\Gamma(G)} \left[ \sum_{t=0}^{(\alpha-1)(n-1)} A_t(\alpha, n-1) \frac{\Gamma(t+s+\alpha)\,\delta^\alpha}{(n+\delta-1)^{t+s+\alpha}} \right.$$

$$\left. + (n-1) \sum_{t=0}^{(\alpha-1)(n-2)} A_t(\alpha, n-2) \sum_{k=0}^{\alpha-1} \left(\frac{\delta^k}{k!}\right) \frac{\Gamma(t+k+\alpha+s)}{(n+\delta-1)^{t+k+\alpha+s}} \right].$$

$$\ldots(16b)$$

If $\alpha = 1$, $n = 2$, (16b) gives

$$\text{Var } v = GH^2/(G-1)^2(G-2)(\delta+1)^2 \qquad \ldots(17)$$

and if $\alpha = 2$, $n = 2$, then (16b) gives

$$E(v) = H\{2(1+\delta^2) + 6\delta\}/(G-1)(\delta+1)^3$$

$$E(v^2) = H^2\{6 + 6\delta^2 + 24\delta\}/(G-1)(G-2)(\delta+1)^4. \qquad \ldots(17b)$$

Table II gives the variance of $x_{(1)} = v$, using (17), (17a) and (17b).

TABLE II

| $\delta$ | 0.5 | 1 | 2 |
|---|---|---|---|
| $\alpha = 1$ | $(0.0988)H^2$ | $(0.0556)H^2$ | $(0.0248)H^2$ |
| $\alpha = 2$ | $(0.3468)H^2$ | $(0.2014)H^2$ | $(0.9868)H^2$ |

(2b) *Maximum* $(x_{(n)} = w)$, $r = n$

If $r = n$, we have the predictve distribution of the maximum $w$ as, from (14), [now last term in (14) vanishes and $m = j$], as,

$$f(w \mid \hat{\theta}) = \left[ \frac{H^G}{\Gamma(\alpha)\,\Gamma(G)} \right] \left[ (n-1)\left(\Omega_0'\right) \sum_{t=0}^{(\alpha-1)j} A_t(\alpha, j) \right.$$

$$\times \left\{ \frac{\Gamma(t+\alpha+G)\, w^{t+\alpha-1}}{[H + w(j+1)]^{t+\alpha+G}} - \sum_{k=0}^{\alpha-1} \left( \frac{\delta^k}{k!} \right) \right.$$

$$\left. \times \frac{\Gamma(t+\alpha+k+G)\, w^{t+k+\alpha-1}}{[H + (j+\delta+1)w]^{t+\alpha+k+G}} \right\} + (\Omega') \sum_{t=0}^{(\alpha-1)j}$$

$$\left. A_t(\alpha, j) \frac{\delta^\alpha\, \Gamma(t+\alpha+G)\, w^{t+\alpha-1}}{[H + (\delta+j)\, w]^{t+\alpha+G}} \right] \qquad \text{...(18)}$$

where $\left(\Omega_0'\right)$, $(\Omega')$ are $(\Omega_0)$ and $(\Omega)$ respectively where $r = n$ and (18) gives

$$\int_0^\infty f(w \mid \hat{\theta})\, dw = 1 = \left[ \frac{1}{\Gamma(\alpha)} \right] \left[ (n-1)\left(\Omega_0'\right) \sum_{t=0}^{(\alpha-1)j} A_t(\alpha, j) \right.$$

$$\left\{ \frac{\Gamma(t+\alpha)}{(j+1)^{t+\alpha}} - \sum_{k=0}^{\alpha-1} \left( \frac{\delta^k}{k!} \right) \frac{\Gamma(t+\alpha+k)}{(j+\delta+1)^{t+\alpha+k}} \right\} + (\Omega') \sum_{t=0}^{(\alpha-1)j}$$

$$\left. A_t(\alpha, j) \frac{\delta^\alpha\, \Gamma(t+\alpha)}{(j+\delta)^{t+\alpha}} \right]. \qquad \text{...(18a)}$$

But

$$\sum_{t=0}^{(\alpha-1)j} A_t(\alpha, j) \frac{\Gamma(t+\alpha)}{\Gamma(\alpha)\,(j+1)^{t+\alpha-1}} = 1 \qquad \text{..(18b)}$$

and also.

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{j+1} = \frac{n!}{\prod_{j=0}^{n} (j+1)}. \qquad \text{...(18c)}$$

Using (18b) for the first term of (18a), we get $(n-1) \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(-1)^j}{(j+1)}$ which is equal to 1. Hence (18a) now gives another identity,

$$\sum_{j=0}^{n-2} (n-1)\binom{n-2}{j}(-1)^j \sum_{t=0}^{(\alpha-1)j} A_t(\alpha,j) \sum_{k=0}^{\alpha-1}\left(\frac{\delta^k}{k!}\right)$$

$$\times \frac{\Gamma(t+k+\alpha)}{\Gamma(\alpha)(j+\delta+1)^{t+k+\alpha}} = \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \sum_{t=0}^{(\alpha-1)j} A_t(\alpha,j)$$

$$\times \frac{\delta^\alpha \, \Gamma(t+\alpha)}{\Gamma(\alpha)(j+\delta)^{t+\alpha}} . \qquad \ldots(19)$$

Now using (15a) with $n = r$ or (18), we get

$$E(W^s) = \frac{H^s \, \Gamma(G-s)}{\Gamma(\alpha)\,\Gamma(G)}\Bigg[(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}(-1)^j \sum_{t=0}^{(\alpha-1)j} A_t(\alpha,j)$$

$$\left\{\frac{\Gamma(t+s+\alpha)}{(j+1)^{t+s+\alpha}} - \sum_{k=0}^{\alpha-1}\left(\frac{\delta^k}{k!}\right)\frac{\Gamma(t+k+s+\alpha)}{(j+\delta+1)^{t+k+s+\alpha}}\right\}$$

$$+\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \sum_{t=0}^{(\alpha-1)j} A_t(\alpha,j)\;\frac{\delta^s\,\Gamma(t+s+\alpha)}{(j+\delta)^{t+s+\alpha}}\Bigg].$$

$$\ldots(19a)$$

From (19a), we have for $\alpha = 1$, $n = 2$, $G = 4$,

$$\text{Var } W = \text{Var } x_{(n)} = \frac{H^2}{6}\left[\left(2+\frac{2}{\delta^2}\right) - \frac{2}{(\delta+1)^2}\right] - \frac{H^2}{9}$$

$$\times\left(\frac{\delta^2+\delta+1}{\delta+\delta^2}\right)^2 \qquad \ldots(19b)$$

and if $\alpha = 2$, $n = 2$, (19a) gives

$$E(W) = \frac{H}{(G-1)}\left[\left(2+\frac{2}{\delta}\right) - \frac{2\delta^2+6\delta+2}{(\delta+1)\,3}\right] \qquad \ldots(20)$$

$$E(W^2) = \frac{H^2}{(G-1)(G-2)}\left[\left(6+\frac{6}{\delta^2}\right) - \frac{6\delta^2+24\delta+6}{(\delta+1)\,4}\right]. \qquad \ldots(20a)$$

Table III gives variance of $x_{(n)} = W$, using (19b), (20), (20a).

TABLE III

| Var $(x)_{(n)}!$ $(G = 4,\ n = 2)$ | | | |
|---|---|---|---|
| $\delta$ | 0.5 | 1 | 2 |
| $\alpha = 1$ | $(0.9136)H^2$ | $(0.3333)H^2$ | $(0.2284)H^2$ |
| $\alpha = 2$ | $(2.2358)H^2$ | $(0.7848)H^2$ | $(0.5590)H^2$ |

### 3. Predictive Distribution (Outlier of Type $\theta + \delta$)

Now (4), looks like

$$f(u \mid \theta) = \left[\frac{a}{\Gamma(\alpha)}\right][(r-1)\{1 - z(\theta u)\}^{r-1}\{z(\theta u)\}^{n-r}\{1 - z[u(\theta$$

$$+ \delta)]\} e^{\theta u}(\theta u)^{\alpha-1}\theta + \{1 - z(\theta u)\}^{r-1}\{z(\theta u)\}^{n-r} e^{-(\theta+\delta)u}$$

$$\times (\theta+\delta)^{\alpha} u^{\alpha-1} + (n-r)\{1 - z(\theta u)\}^{r-1}\{z(\theta u)\}^{n-r-1}$$

$$\times \{z[u(\theta+\delta)]\} e^{-\theta u}(\theta u)^{\alpha-1}\theta] \qquad \text{...(21)}$$

$$= [a/\Gamma(\alpha)][(r-1)(\Omega_0)\{z(\theta u)\}^m\{1 - z[u(\theta+\delta)]\} e^{-\theta u}(\theta u)^{\alpha-1}\theta$$

$$+ (\Omega)\{z(\theta u)\}^m e^{-(\theta+\delta)u}(\theta+\delta)^{\alpha} u^{\alpha-1}$$

$$+ (n-r)(\Omega)\{z(\theta u)\}^{m-1}\{z[u(\theta+\delta)]\} e^{-\theta u}(\theta_{\mu})^{\alpha-1}\theta] \qquad \text{..(21a)}$$

$$= [a/\Gamma(\alpha)][(r-1)(\Omega_0)\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\ \{\phi(m+1)(\theta u)^{t+\alpha-1}\theta$$

$$-\phi(m+2)(e^{-\delta u})\sum_{k=0}^{\alpha-1}\sum_{s=0}^{k}\binom{k}{s}\left(\frac{\delta^{k-s}}{k!}\right)(\theta u)^{t+\alpha-1}$$

$$\times u^k \theta^{s+1}\} + (\Omega)\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\,\phi(m+1)\, e^{-\delta u}\sum_{s=0}^{\alpha}$$

$$\binom{\alpha}{s}\delta^{\alpha-s} u^{t+\alpha-1}\theta^{s+t} + (n-r)(\Omega)\sum_{t=9}^{(\alpha-1)(m-1)} A_t(\alpha, m-1)$$

$$\phi(m+1)\sum_{k=0}^{\alpha-1}\sum_{s=0}^{k}\binom{k}{s} e^{-\delta u}(\delta^{k-s}/k!)(\theta u)^{t+\alpha-1} u^k \theta^{s+1}]. \qquad \text{...(22)}$$

From (12) and (22), we get the predictive distribution of the $r$th order statistics, as

$$\int_0^{\infty} f(u \mid \hat{\theta}) f(\theta \mid \hat{\theta})\, d\theta = f(u \mid \hat{\theta})$$

$$= \left[\frac{aH^G}{\Gamma(\alpha)\Gamma(G)}\right]\left[(r-1)(\Omega_0)\sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\left\{\frac{u^{t+\alpha-1}\Gamma(t+\alpha+G)}{[H+(m+1)u]^{t+\alpha+G}}\right.\right.$$

(equation continued on p. 868)

$$- \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s} \left( \frac{\delta^{k-s}}{k!} \right) \frac{e^{-\delta u}\, u^{t+k+\alpha-1}\, \Gamma(t+\alpha+s+G)}{[H+(m+2)u]^{t+\alpha+s+G}} \Bigg\}$$

$$+ (\Omega) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m)\, e^{-\delta u} \sum_{s=0}^{\alpha} \binom{\alpha}{s} \frac{\delta^{\alpha-s}\, u^{t+\alpha-1}(t+s+G)}{[H+(m+1)u]^{t+s+G}}$$

$$+ (n-r)(\Omega) \sum_{t=0}^{(\alpha-1)(m-1)} A_t(\alpha, m-1) \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s} \left( \frac{\delta^{k-s}}{k!} \right)$$

$$u^{t+k+\alpha-1} \frac{(t+s+G)\, e^{-\delta u}}{[H+(m+1)u]^{t+s+G+\alpha}} \Bigg]. \qquad \ldots(23)$$

Now, from Erdelyi *et al.*[4], we have the hypergeometric function

$$\psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-tx}\, t^{a-1}}{(1+t)^{a-c+1}}\, dt. \qquad \ldots(24)$$

Using (24) in (23) we have

$$\int_0^\infty f(u \mid \hat{\theta})\, du = (a) \Bigg[ (r-1)(\Omega_0) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m) \Bigg\{ \frac{\Gamma(t+\alpha)}{\Gamma(\alpha)(m+1)^{t+\alpha}}$$

$$- \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s} \frac{(H\delta)^{k-s}}{k!}$$

$$\times \left( \frac{\Gamma(t+s+\alpha+G)\, \Gamma(t+k+\alpha)}{\Gamma(G)\, \Gamma(\alpha)\, (m+2)^{t+k+\alpha}} \right)$$

$$\times \psi\left( t+k+\alpha, k-s-G+1; \frac{H\delta}{m+2} \right) \Bigg\}$$

$$+ (\Omega) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m) \sum_{s=0}^{\alpha} \binom{\alpha}{s} (H\delta)^{\alpha-s}$$

$$\times \left( \frac{\Gamma(t+\alpha)\, \Gamma(G+t+s)}{\Gamma(\alpha)\, \Gamma(G)\, (m+1)^{t+\alpha}} \right) \times \psi\Big( t+\alpha,$$

$$\alpha+1-s-G; \frac{H\delta}{m+1} \Big) + (n-r)(\Omega) \sum_{t=0}^{(\alpha-1)(m-1)}$$

(*equation continued on p. 869*)

$$A_t(\alpha, m-1) \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s} \frac{\delta H)^{k-s}}{k!}$$

$$\times \left( \frac{\Gamma(t+k+\alpha)\,\Gamma(t+s+\alpha+G)}{\Gamma(\alpha)\,\Gamma(G)\,(m+1)^{t+k+\alpha}} \right)$$

$$\times \psi \left( t+k+\alpha, k+1-s-G; \frac{H\delta}{m+1} \right) \Big] . \quad \ldots(25)$$

Now, again from Erdelyi *et al.*[4], one has

$$\psi(a, c; x) = x^{1-c} \; \psi(a-c+1, 2-c; x) \quad \ldots(26)$$

first term in (25) is equal to 1 using (18b), (18c) and hence (25) looks like, using (26),

$$\int_0^\infty f\,(u \mid \hat{\theta})\, du = 1 - \Big\{ a\,(r-1)\,(\Omega_0) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m) \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s}$$

$$\times \frac{(H\delta)^G}{k!} \left[ \frac{\Gamma(t+s+\alpha+G)\,\Gamma(t+k+\alpha)}{\Gamma(\alpha)\,\Gamma(G)\; m+2)^{t+s+\alpha+G}} \right]$$

$$\times \psi \left( t+s+\alpha+G, s+G+1-k; \; \frac{(H\delta)}{m+2} \right)$$

$$+ (a)\,(\Omega) \sum_{t=0}^{(\alpha-1)m} A_t(\alpha, m) \sum_{s=0}^{\alpha} \binom{\alpha}{s}$$

$$\times \left[ \frac{(H\delta)^G\,\Gamma(t+s+G)\,\Gamma(t+\alpha)}{\Gamma(\alpha)\,\Gamma(G)\,(m+1)^{G+s+t}} \right] \psi\,(t+s\,G,$$

$$G+s+1-\alpha; \frac{H\delta}{M+1} \Big) + a\,(n-r)\,(\Omega) \sum_{t=0}^{(\alpha-1)(m-1)}$$

$$\times A_t(\alpha, m-1) \sum_{k=0}^{\alpha-1} \sum_{s=0}^{k} \binom{k}{s} \frac{(\delta H)^G}{k!}$$

$$\times \left( \frac{\Gamma(t+k+\alpha)\,\Gamma(t+s+\alpha+G)}{\Gamma(\alpha)\,\Gamma(G)\,(m+1)^{t+s+\alpha+G}} \right)$$

$$\psi \left( t+s+\alpha+G, G+1+s-k; \frac{H\delta}{m+1} \right) \Big\} . \quad \ldots(27)$$

## 4. Comments

1. In this paper, $\alpha$ is taken as a positive integer, which is not a serious restriction. Otherwise, coefficients $A_t\ (\alpha, m)$ are not valid.

2. Single outlier of type $\theta\delta$ or of type $\theta + \delta$ is dealt with. For the case of multiple outliers, analysis will be more heavy.

3. In both types of outlier cases, one has to have the tables of coefficients $A_t(\alpha, m)$ to find the probability $p\ (u \geqslant u_0)$ and these can easily be generated from the relation (7).

4. One can easily evaluate means and variances from (23) again similar to (14), in this case, using the hypergeometric functions.

5. Thou gh the expressions in the paper seem complex; on the computers, sums can be easily handled for any values of $\alpha$ and $n$.

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# ON THE FORMS OF $n$ FOR WHICH $\varphi(n) \mid n-1$

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The forms of $n$ for which $\varphi(n) \mid n-1$, where $\varphi(n)$ is the Euler totient function are obtained in this paper.

§1. Let $\varphi(n)$ denote the Euler totient function. It is obvious that

$$\varphi(n) \mid n-1 \qquad \ldots(1.1)$$

if $n$ is a prime. Lehmer[2] asked whether there is any composite number $n$ for which (1.1) holds. This unsolved problem, which is as deep as the odd perfect number problem, has several partial answers. To mention few, Lehmer[2] has proved that any such $n$, if exists, is odd, squarefree and that if $p, q$ are distinct prime factors of $n$ then $p \not\equiv 1 \pmod q$. More recently Cohen and Hagis[1] established that such $n$ must have at least 14 distinct prime factors, while the authors[3] have proved that certain types of composite numbers can not satisfy (1.1). For further details and analogues of the problem we refer to Subbarao and Prasad[4] and Prasad and Subbarao[5].

In this note we find the forms in which the composite $n$ satisfying (1.1) should be, if exists (Theorem 2.1).

§2. In all that follows $n$ denotes a composite number for which (1.1) is true so that, by the above paragraph we can write

$$n = p_1 p_2 p_3 \ldots p_r \qquad \ldots(2.1)$$

where $p_1 < p_2 < \ldots < p_r$ are odd primes ;

$$p_i \not\equiv 1 \pmod{p_j} \text{ for } i \neq j \, ; \qquad \ldots(2.2)$$

and if $w(n)$ is the number of distinct prime factors of $n$ then

$$r = w(n) \geqslant 14. \qquad \ldots(2.3)$$

Let $s$ denote the number of $p_i \equiv -1 \pmod 3$. Then we have $s \leqslant r$ and the equality holds if and only if $p_i \equiv -1 \pmod 3$ for $1 \leqslant i \leqslant r$. Moreover if $p_1 = 3$ then by (2.2), we have $s = r-1$.

*Theorem* 2.1—Suppose $n$ is as in (2.1)

(i) If $p_1 = 3$ then $n$ is of the form $2^{14}\,3^2\,m + 81921$ or $2^{14}\,3^2m + 131073$ according as $s$ is even or odd.

(ii) If $p_1 > 3$ then $n$ is of the form $2^{14}\,3m + 1$ or $2^{14}\,3m + 65537$ according as $s$ is even or odd.

PROOF : Since $\varphi(n) = (p_1 - 1)(p_2 - 1) \ldots (p_r - 1) \equiv O \pmod{2^r}$ we get, by (2.3) and (1.1), that

$$n - 1 \equiv O \pmod{2^{14}}. \qquad \ldots(2.4)$$

(i) Suppose $p_1 = 3$. Then by (2.2), $p_i \equiv -1 \pmod 3$ for $2 \leqslant i \leqslant r$ so that $n/3 = p_2\, p_3 \ldots p_r \equiv (-1)^s \pmod 3$ and therefore

$$n = 9l + 3(-1)^s \text{ for some integer } l. \qquad \ldots(2.5)$$

If $s$ is even, (2.5) gives $n = 9l + 3$ and (2.4) shows that $l$ satisfies the congruence $9l + 2 \equiv O \pmod{2^{14}}$. That is, $l \equiv 9102 \pmod{2^{14}}$. Writing $l = 2^{14}m + 9102$, we get $n = 2^{14}\,3^2m + 81921$

If $s$ is odd, (2.5) gives $n = 9l - 3$ for some integer $l$ and by (2.4), $l$ must be such that $9l \equiv 4 \pmod{2^{14}}$. That is $l \equiv 14564 \pmod{2^{14}}$ and hence $n = 2^{14}\,3^2\,m + 131073$ for some integer $m$.

(ii) If $p_1 > 3$ then, by (2.1), $n \equiv (-1)^s \pmod 3$.

Now if $s$ is even, then $n \equiv 1 \pmod 3$ which together with (2.4) implies $n \equiv 1 \pmod{2^{14}3}$, proving the first part of (ii).

When $s$ is odd, $n = 3u - 1$ for some integer $u$, which must satisfy the congruence $3u \equiv 2 \pmod{2^{14}}$, by (2.4). That is $u \equiv 21846 \pmod{2^{14}}$ and hence $n = 2^{14}.\,3m + 65537$ for some integer $m$.

*Theorem* 2.2—Suppose $n$ is as in (2.1) with $p_1 > 3$.

(i) If $s < r$ then $n$ is of the form $2^{14}\,3m + 1$.

(ii) When $s = r$ we have $n$ is of the form $2^{14}\,3m + 1$ if and only if $s$ is even.

PROOF : (i) If $s < r$ then $p_i \equiv 1 \pmod 3$ for at least one $i$ so that $\varphi(n) \equiv O \pmod 3$ and hence $n \equiv 1 \pmod 3$. This together with (2.4) proves the first part.

(ii) Suppose $s = r$. If $s$ is even, then, by (2.1), $n \equiv 1 \pmod 3$ which combined with (2.4) shows that $n$ is of the form $2^{14}\,3m + 1$.

Conversely if $n = 2^{14}\,3m + 1$ then $n \equiv 1 \pmod 3$ and by (2.1), $n \equiv (-1)^s \pmod 3$ so that $(-1)^s \equiv 1 \pmod 3$ proving $s$ is even.

*Corollary* 2.3—Suppose $n$ is as in (2.1) with $p_1 > 3$. Then $n$ is of the form $2^{14}\,3m + 65537$ if and only if $s = r$ and $s$ is odd.

PROOF : Follows from (ii) of Theorem 2.1 and 2.2.

*Remark* : Cohen and Hagis[1] have proved that $n > 10^{20}$ while Subbarao and one of the authors[5] showed that $n < (r-1)^{2^{r-1}}$, where $r = w(n)$. Hence if $r$ is an integer such that there is no $m$ in the above forms with $10^{20} < m < (r-1)^{2^{r-1}}$ then $w(n) > r$, for $n$ satisfying (1.1).

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# THE INJECTIVE HULL OF A MODULE WITH FGD

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The concepts linearly independent and $u$-element are introduced and it is proved that if $R$ is a ring with unity, $M$ is an unitary $R$-module, and $R$ has finite Goldie dimension then $a \in M$ is an $u$-element if and only if either $(0 : a)$ is an essential and $E$-irreducible submodule or $(0 : a)$ is a maximal complement submodule of the $R$-module $R$. It is found that there exists a set consisting of some indecomposable injective $R$-modules such that the injective hull of every $R$-module with finite Goldie dimension is isomorphic with a direct sum of a finite number of elements from the set.

## Introduction

In Section 1 the concept 'linearly independent' in vector spaces is extended to modules. In this connection the concepts linearly independent, $u$-element, $u$-linearly independent and $B$-linearly independent are introduced and the relation between some of these concepts is observed. In Section 2 we obtain an equivalent condition for an element $a \in M$ to be an $u$-element. Section 3 concentrates on injective hulls and three results are obtained namely Theorems 3.1, 3.2 and 3.3.

Throughout $R$ denotes a ring (not necessarily commutative) and $M$ a left $R$-module. '$K \leqslant_s M$' means '$K$ is a submodule of $M$'. As in Goldie[4], Reddy and Satyanarayana[8] and Sharpe and Vamaos[7], we shall use the following terminology. $K \leqslant_s M$ is called essential in $M$ (or $M$ is an essential extension of $K$) (denoted by $K \leqslant_e M$) if $K \cap A = (0)$ for any other submodule $A$ of $M$, implies $A = (0)$. $M$ has finite Goldie dimension (abbr. FGD) if $M$ does not contain a direct sum of an infinite number of non-zero submodules. Equivalently, $M$ has *FGD* if for any strictly increasing sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ of submodules of $M$, there is an integer $i$ such that for every $k \geqslant i$, $H_k \leqslant_e H_{k+1}$. $M$ is uniform if every non-zero submodule of $M$ is essential in $M$. Goldie[4] has proved that in any module $M$ with *FGD*, there exists non-zero uniform submodules $U_i$, $1 \leqslant i \leqslant n$ whose sum is direct and essential in $M$. The number '$n$' is independent of the uniform submodules. This number $n$ is called the Goldie dimension of $M$ and denoted by dim $M$. Furthermore, if $A \leqslant_s M$ and $K$ is a submodule of $M$ maximal with respect to $K \cap A = (0)$, then we say that $K$ is a complement of $A$ (or a complement in $M$ or complement submodule in $M$) (denoted

by $K \leqslant_c M$). $K \leqslant_c M$ is said to be a maximal complement submodule if (i) $K \neq M$; and (ii) $K \subseteq L \subseteq M$ and $L \leqslant_c M$, imply $K = L$ or $L = M$. $H \leqslant_s M$ is said to be irreducible if $H = K \cap J$, $K \leqslant_s M$ and $J \leqslant_s M$, imply $H = K$ or $H = J$. $H \leqslant_s M$ is said to be $E$-irreducible if $H = K \cap J$, $K \leqslant_s M$, $J \leqslant_s M$ and $H \leqslant_e K$, imply $H = K$ or $H = J$. $M$ is said to be indecomposable if $M \neq (0)$ and the only direct summands of $M$ are (0) and itself. The symbol 'iff' means 'if and only if'. The direct sum of $R$-modules $K_i$, $1 \leqslant i \leqslant n$ is denoted by $\overset{n}{\underset{i=1}{\oplus}} K_i$ and the injective hull of an $R$-module $A$ is denoted by $E(A)$. If $K_i = K$, $1 \leqslant i \leqslant n$ then we write $K^{(n)}$ for $\overset{n}{\underset{i=1}{\oplus}} K_i$. Throughout this paper the word 'module' means 'module with $FGD$'. $(a)$ and $(X)$ denote the submodules generated by an element '$a$' and a set '$X$' respectively. From Goldie[4] we list the following results which are to be used frequently:

*Theorem* 0.1—(a) $(0) \neq K \leqslant_s M$ implies $K$ contains an uniform submodule.

(b) (i) If $k = \dim M$ and $U_i$, $1 \leqslant i \leqslant n$ are uniform submodules whose sum is direct then $n \leqslant k$.

(ii) $\dim M = n$ iff there exist uniform submodules $U_i$, $1 \leqslant i \leqslant n$ whose sum is direct and essential in $M$.

(c) Suppose $K_i$, $1 \leqslant i \leqslant n$ are submodules whose sum is direct and $L_i \leqslant_s M$, $1 \leqslant i \leqslant n$ such that $L_i \subseteq K_i$ then $L_i \leqslant_e K_i$ for each $1 \leqslant i \leqslant n$ iff $\overset{n}{\underset{i=1}{\oplus}} L_i$ is essential in $\overset{n}{\underset{i=1}{\oplus}} K_i$.

(d) For $K \leqslant_s M$ we have $K \leqslant_e M$ iff $\dim K = \dim M$.

(e) (i) $K \leqslant_c M$ implies $\dim (M/K) = \dim M - \dim K$.

(ii) If $K$ is a maximal complement submodule of $M$ then $\dim K = \dim M - 1$.

## Section 1

In this section we list the usual results about a module with FGD in terms of linear independence of a set of elements. Most of the results are straight forward modifications of known results and the proofs will not be given in detail. We illustrate this in the proof of Result 1.2. The proofs for some results are omitted when they are easy.

*Definition* 1.1—$0 \neq a_i \in M$, $1 \leqslant i \leqslant n$, are said to be linearly independent if the sum of $(a_i)$, $1 \leqslant i \leqslant n$ is direct. If $a_i$, $1 \leqslant i \leqslant n$ are linearly independent then the set $\{a_i / 1 \leqslant i \leqslant n\}$ is called a linearly independent set in $M$. If $X$ is not linearly independent we say it is linearly dependent.

*Result* 1.1—If $a_i$, $1 \leqslant i \leqslant n$ are linearly independent elements then $n \leqslant \dim M$.

PROOF : By Theorem 0.1 (*a*), there exist uniform submodules $U_i$, $1 \leqslant i \leqslant n$ such that $U_i \subseteq (a_i)$. Since $a_i$, $1 \leqslant i \leqslant n$ are linearly independent the sum $\sum_{i=1}^{n} (a_i)$ is direct and so $\sum_{i=1}^{n} U_i$ is also a direct sum. By Theorem 0.1 (b) (i), we have $n \leqslant \dim M$.

*Result* 1.2—$\dim M$ is equal to the least upper bound of the set $A$ where $A = \{n/\text{'}n\text{'}$ is a natural number and there exist $a_i \in M$, $1 \leqslant i \leqslant n$ such that $a_i$, $1 \leqslant i \leqslant n$ are linearly independent$\}$.

PROOF : By Result 1.1, $\dim M$ is an upper bound for $A$. Write $k = \dim M$. Using Theorem 0.1 (b) (ii), it follows that $k \in A$.

*Result* 1.3—If $n = \dim M$ and $\{a_i/1 \leqslant i \leqslant n\}$ is a linearly independent set then each $(a_i)$ is an uniform submodule.

PROOF : Suppose there exist $(0) \neq B \leqslant_s (a_k)$, $(0) \neq D \leqslant_s (a_k)$ such that $B \cap D = (0)$ for some $k$, $1 \leqslant k \leqslant n$. Let $0 \neq b \in B$ and $0 \neq d \in D$. Since $(b) \cap (d) = (0)$, we have $a_1, \ldots a_{k-1}, a_{k+1}, \ldots a_n, b, d$ are linearly independent, contradicting Result 1 2.

*Definitions* 1.2—$0 \neq a \in M$ is called an $u$-element if $(a)$ is an uniform submodule. The elements $a_i \in M$, $1 \leqslant i \leqslant n$ are said to be $u$-linearly independent elements if they are linearly independent and each $a_i$ is an $u$-element. If $a_i \in M$, $1 \leqslant i \leqslant n$ are $u$-linearly independent, then the set $X = \{a_i/1 \leqslant i \leqslant n\}$ is called an $u$-linearly independent set. We say that a ($u$-) linearly independent set $X = \{a_i/1 \leqslant i \leqslant n\}$ in $M$ is a maximal ($u$-) linearly independentset if $X \cup \{b\}$ is ($u$-)-linearly dependent for each $b \in M \setminus X$ (see P. 84 of Fuchs[3]).

*Note* 1.1 : In view of Results 1 2 and 1.3, if $\dim M = n$ then for all linearly independent sets $X \subseteq M$ with $|X| = n$ we have (i) $X$ is a maximal linearly independent set; (ii) $X$ is an $u$-linearly independent set; and (iii) $(0) \neq A \leqslant_s M \Rightarrow A$ contains an $u$-element (follows from Theorem 0.1 (a)).

*Result* 1.4 (i)—If $b_i$, $1 \leqslant i \leqslant n$ are linearly independent, then there exist $u$-elements $a_i \in (b_i)$, $1 \leqslant i \leqslant n$ such that $a_i$, $1 \leqslant i \leqslant n$ are $u$-linearly independent. (ii) If $(0) \neq K \leqslant_s M$, then there exist an $u$-linearly independent set $X = \{a_i/1 \leqslant i \leqslant n\}$ such that $(X) = \bigoplus_{i=1}^{n} (a_i)$ is essential in $K$. Moreover, $n = \dim K$.

PROOF : (i) follows from Note 1.1 (iii). If $n = \dim K$ then there exist uniform submodules $U_i$, $1 \leqslant i \leqslant n$ whose sum is direct and essential in $K$. Let $0 \neq a_i \in U_i$ for $1 \leqslant i \leqslant n$. Now the result follows with $X = \{a_i/1 \leqslant i \leqslant n\}$.

*Result* 1.5—dim $M = n$ iff there exists a maximal $u$-linearly independent set $S$ with $|S| = n$.

PROOF : Proof follows from Theorem 0.1 (b) (ii) and the fact that a set $\{a_i / 1 \leqslant i \leqslant n\} \subseteq M$ is maximally linearly independent iff $\overset{n}{\underset{i=1}{\oplus}} (a_i) \leqslant_e M$.

*Definition* 1.3—A maximal $u$-linearly independent set $X = \{a_i / 1 \leqslant i \leqslant n\} \subseteq M$ is called a basis for $M$.

*Remark* 1.1 : If $S$ is the set of all $u$-elements of $M$ then there exists $\{b_i / 1 \leqslant i \leqslant k\} \subseteq S$ which forms a basis for $M$. Also $k = \dim M$ iff $M$ has a basis containing $k$ elements.

*Definitions* 1.4—$0 \neq a_i \in M$, $1 \leqslant i \leqslant n$ are said to be $B$-linearly dependent if there exist $r_i \in R$, $1 \leqslant i \leqslant n$ not all of them zero such that $r_1 a_1 + \ldots + r_n a_n = 0$. If $a_i$, $1 \leqslant i \leqslant n$ are not $B$-linearly dependent then they are said to be $B$-linearly independent.

*Result* 1.6—If $R$ is a ring with unity, $M$ is an unitary $R$-module and $a_i$, $1 \leqslant i \leqslant n$ are $B$-linearly independent then they are linearly independent.

*Definitions* 1.4—$a \in M$ is said to be a torsion element of $M$ if there exists $0 \neq r \in R$ such that $ra = 0$. We denote the set of all torsion elements of $M$ by $T(M)$. If $T(M) = M$ $(T(M)) = (0))$ then $M$ is said to be a torsion (torsion-free) module.

*Result* 1.7—If $M$ is torsion-free then $a_i$, $1 \leqslant i \leqslant n$ are linearly independent implies they are $B$-linearly independent.

*Corollary*—If $M$ is a torsion-free module then $a_i$, $1 \leqslant i \leqslant n$ are $u$-linearly independent iff they are $B$-linearly independent and each $a_i$ is an $u$-element.

*Definition* 1.5—If $S = \{a_i \in M / 1 \leqslant i \leqslant n\}$ then the set $L(S) = \{r_1 a_1 + \ldots + r_n a_n / r_i \in R$ for $1 \leqslant i \leqslant n\}$ is called the linear span of $S$.

*Note* 1.2 : Suppose $S = \{a_i \in M / 1 \leqslant i \leqslant n\}$. Then (i) $L(S) \leqslant_s M$; (ii) if $R$ is a ring with unity and $M$ is an unitary $R$-module then $L(S) = (S)$; and (iii) if $S$ is a $B$-linearly independent set then every element in $L(S)$ has a unique representation as $r_1 a_1 + \ldots + r_n a_n$ with $r_i \in R$ for $1 \leqslant i \leqslant n$.

*Result* 1.8—If $R$ is a skew-field then $X = \{a_i / 1 \leqslant i \leqslant n\}$ is a basis for $M$ iff $X$ is linearly independent and $M = L(X)$.

*Remark* 1.3 : If $M$ is a vector space, the concepts '$B$-linearly independent', '$u$-linearly independent' and 'linearly independent' are one and the same. In view of Result 1.8, our concept of 'basis' is the usual concept for a vector space.

*Result* 1.9—If $K$ is a maximal complement then there exists an uniform submodule $U$ such that $K \cap L = (0)$ and $K + L \leqslant_e M$.

PROOF : By Theorem 0.1 (*e*), $\dim K = \dim M - 1$ and by Theorem 0.1 (d), $K$ is not essential in $M$. Therefore there exists $(0) \neq L \leqslant_s M$ such that $K \cap L = (0)$. By Exercise 2 (3), page 294 of Anderson and Fuller[1], $\dim (K + L) = \dim K + \dim L \geqslant (\dim M - 1) + 1 = \dim M$. This shows that $K + L \leqslant_e M$. Now $\dim L = \dim M - \dim K = 1$ imply $L$ is uniform.

*Result* 1.10—If $K$ is an essential and $E$-irreducible submodule of $M$ then $K$ is irreducible.

PROOF : Let $I$ and $J$ are two submodules of $M$ properly containing $K$ such that $K = I \cap J$. Since $K \leqslant_e M$ and $K \subseteq I \subseteq M$ we have $K \leqslant_e I$, a contradiction to the fact that $K$ is $E$-irreducible.

SECTION 2

*Notation* 2.1—The set of all $u$-elements together with '0' is denoted by $S(M)$. When we go through the following result, we will learn that in general, $S(M)$ may not be a submodule of $M$. If $x \in M$ then the left ideal $\{r \in R / rx = 0\}$ is denoted by $(0 : x)$. Throughout this section $R$ is a ring with unity and $M$ is an unitary $R$-module.

*Result* 2.1—If $x, y$ are two $u$-elements in $M$ satisfying $Rx \cap Ry = (0)$, $(0 : x) \cap (0 : y) = (0)$ and $(0 : x) + (0 : y) = R$ then $x + y$ is not an $u$-element

PROOF : Write $z = x + y$. Since the sum $Rx + Ry$ is direct and $Rz = [(0 : x) + (0 : y)]\, z = (0 : x)\, y + (0 : y)\, x = Rx + Ry$ we have that $z$ is not an $u$-element.

*Example* 2.1—In $Z_6$ (the ring of integers modulo 6) the elements 2 and 3 are $u$-elements but by Result 2.1, their sum 5 is not an $u$-element. Thus, in general, $S(M)$ need not be a submodule of $M$.

*Result* 2.2—If $M$ is torsion-free then $S(M) \leqslant_s M$

PROOF : Let $x, y \in S(M)$. Suppose $x \neq 0$ and $x - y \neq 0$. The isomorphism $f : Rx \to R(x - y)$; $f(rx) = rx - ry$ shows that $x - y \in S(M)$. One can easily shows that $ra \in S(M)$ for all $r \in R$ and $a \in S(M)$.

*Result* 2.3—If $S(M) \cap T(M) = (0)$ and $U$ is an uniform submodule then $U$ contains an isomorphic copy of $R$. Moreover $R$ is an uniform $R$-module.

PROOF : Let $0 \neq a \in U$. By the assumed condition, the epimorphism $f : R \to Ra$; $f(r) = ra$, is an isomorphism. Now $R \cong Ra \subseteq U$ implies that $R$ is an uniform $R$-module.

*Corollary*—If $S(M) \cap T(M) = (0)$ and $\dim M = n$ then there exists $A \leqslant_e M$ such that $A \cong R^{(n)}$.

*Lemma* 2.1—(i) If $J \leqslant_s K$ then $K/J$ is uniform iff $J$ is irreducible. (ii) If $a \in M$, then '$a$' is an $u$-element iff $(0 : a)$ is an irreducible submodule of the $R$-module $R$.

PROOF : (i) Routine. (ii) follows from (i) and the fact $Ra \cong R/(0 : a)$.

*Theorem* 2.1—If $R$ has FGD and $a \in M$ then $a$ is an $u$-element iff either $(0 : a)$ is an essential and $E$-irreducible submodule or $(0 : a)$ is a maximal complement submodule of $R$. In this case

$$\dim((0 : a)) \geqslant \dim R - 1.$$

PROOF : If $a$ is an $u$-element then by Lemma 2.1, $(0 : a)$ is an irreducible submodule and so $(0 : a)$ is an $E$-irreducible submodule of $R$. If $(0 : a)$ is not an essential submodule of $R$ then there exists an uniform submodule $K$ of $R$ such that $(0 : a) \cap K = (0)$. Let $L$ be a maximal element in the set $\mathscr{P} = \{Z/Z$ is a submodule of the $R$-module $R$ such that $Z \supseteq (0 : a)$ and $Z \cap K = (0)\}$. Clearly $L$ is a complement submodule in $R$. We claim that $L = (0 : a)$. If not, we have two non-zero submodules $L/(0 : a)$ and $(K + (0 : a))/(0 : a)$ whose intersection is zero, a contradiction to the uniformity of $R/(0 : a) \cong Ra$. Thus $(0 : a)$ is a complement submodule. Since $K$ is uniform we have $(0 : a)$ is a maximal complement submodule.

*Converse—Case* (i) : If $(0 : a)$ is a maximal complement submodule, then by Theorem 0.1 (e), $\dim(R/(0 : a)) = \dim R - \dim(0 : a)$ and $\dim(0 : a) = \dim R - 1$. So $\dim(R/(0 : a)) = 1$. This implying $Ra \cong R/(0 : a)$ is an uniform $R$-module and hence '$a$' is an $u$-element.

*Case* (ii) : By Result 1 10, $(0, a)$ is an irreducible submodule and by Lemma 2.1, '$a$' is an $u$-element. Now it is clear that (i) if $(0 : a)$ is essential in $R$, then $\dim(0 : a) = \dim R > \dim R - 1$, and (ii) if $(0 : a)$ is a maximal complement in $R$, then $\dim(0 : a) = \dim R - 1$, according to Theorem 0.1 (e).

## SECTION 3

*Lemma* 3.1—If $K$ is an injective module then the following conditions are equivalent: (i) $K = E(U)$ for some uniform module $U$; (ii) $K$ is an indecomposable injective module; (iii) $\dim K = 1$; (iv) The submodule (0) of $K$ is irreducible; and (v) The submodule (0) of $K$ is $E$-irreducible.

Although the next Lemma is well known (see p. 294 of Anderson and Fuller[1]) we give the proof as we will refer to the proof later.

*Lemma* 3.2—$n = \dim M$ iff there exist indecomposable injective modules $K_i$, $1 \leqslant i \leqslant n$ such that $E(M) = \bigoplus_{i=1}^{n} K_i$.

PROOF : For necessity, since $n = \dim M$, there exist uniform submodules $U_i$, $1 \leqslant i \leqslant n$ whose sum is direct and essential in $M$. Write $K_i = E(U_i)$ for $1 \leqslant i \leqslant n$. Now by Lemma 3.1, each $K_i$ is an indecomposable injective module and by Proposition 2.23[7], $\overset{n}{\underset{i=1}{\oplus}} K_i = E(\overset{n}{\underset{i=1}{\oplus}} U_i)$. By Proposition 2.22[7], $E(\overset{n}{\underset{i=1}{\oplus}} U_i) = E(M)$ and so $E(M) = \overset{n}{\underset{i=1}{\oplus}} K_i$. Sufficiency follows readily.

*Theorem* 3.1—Suppose $\mathcal{M}$ is an isomorphism closed class of $R$-modules such that there exists an indecomposable injective $K \in \mathcal{M}$; and if $A, B \in \mathcal{M}$, then $A + B \in \mathcal{M}$. Then the following two conditions are equivalent: (i) $A \cong B \Leftrightarrow \dim A = \dim B$, for $A, B \in \mathcal{M}$ and (ii) $A \in \mathcal{M} \Rightarrow A \cong K^{(n)}$ for some $n \in \mathbb{N}$ (in other words, $\mathcal{M} = \{\overset{n}{\underset{i=1}{\oplus}} K_i / K_i \cong K$ for $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}\}$).

PROOF : (i) $\Rightarrow$ (ii) : Let $A \in \mathcal{M}$. Since $\dim A = \dim E(A)$, we have $A \cong E(A)$ and so $A = E(A)$. By Lemma 3.2, there exist indecomposable injective modules $K_i$, $1 \leqslant i \leqslant n$ such that $E(A) = \overset{n}{\underset{i=1}{\oplus}} K_i$. Since each $K_i$ is an indecomposable injective module we have $\dim K_i = \dim K$ and so $K_i \cong K$ for each $i$. Thus $A \cong K^{(n)}$ and hence we have (ii). (ii) $\Rightarrow$ (i) : Let $A, B \in \mathcal{M}$. If $\dim A = \dim B$ then $A = \overset{p}{\underset{i=1}{\oplus}} K_i$ and $B = \overset{q}{\underset{j=1}{\oplus}} L_j$ where $K_i \cong K \cong L_j$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$ Now by Lemma 3.2, $p = \dim A = \dim B = q$ and hence $A \cong B$.

*Theorem* 3.2—Let $R$ be a ring with unity and $M$ an unitary $R$-module. If $R$ has FGD, $\dim M = n$ and $S(M) \cap T(M) = (0)$ then $E(M) = \overset{n}{\underset{i=1}{\oplus}} K_i$ where $K_i \cong E(R)$.

PROOF : By Corollary of Result 2.3, there exists $A \leqslant_e M$ such that $A \cong R^{(n)}$. Let $K = E(R)$. Then by Proposition 2.23[7], $K^{(n)} = E(R^{(n)})$ and so $K^{(n)} \cong E(A)$. Since $E(A) \cong K^{(n)}$ there exist injective modules $K_i$, $1 \leqslant i \leqslant n$ such that $K_i \cong K$ and $E(A) = \overset{n}{\underset{i=1}{\oplus}} K_i$. Since $A \leqslant_e M$ by Proposition 2.24[7] we have $E(M) = \overset{n}{\underset{i=1}{\oplus}} K_i$.

In Result 2.3 (Corollary) the $R$-module $R$ is uniform. From this fact we can get the following Corollary:

*Corollary* 1—If $S(M) \cap T(M) = (0)$ then $M$ can be embedded essentially in $E(R)^{(n)}$ where $n = \dim M$ and $E(R)$ is an indecomposable injective module.

*Corollary* 2—If $\dim M = n$ and $S(M) \cap T(M) = (0)$ then

$$\{E(A)/(0) \neq A \leqslant_s M\} = \{\bigoplus_{i=1}^{n} K_i / K_i \cong E(R) \text{ for } 1 \leqslant j \leqslant n \text{ and } n \in \mathbb{N}\}.$$

*Lemma* 3.4—If $K$ is injective, $U$ is uniform and $U \cap K \neq (0)$ then $U \subseteq K$.

Proof : If $Z = U \cap K$ then $Z \leqslant_e U$ and so $U \subseteq E(U) = E(Z) \subseteq K$.

Consider $X$, the set of all $u$-elements in $R$ and define an equivalence relation '$\sim$' on $X$ as $a \sim b \Leftrightarrow (a) \cap (b) \neq (0)$ for all $a, b \in X$. Suppose the disjoint equivalence classes are $\{S_i\}_{i \in I}$. Write $X_i = S_i \cup \{0\}$.

*Lemma* 3.5—For each $i \in I$, $X_i$ is an uniform submodule of $R$.

Proof : Let $a \in S_i$ and $K = E((a))$. Since $(a)$ is uniform, $K$ is also uniform. $x \in S_i$ implies $(0) \neq (a) \cap (x) \subseteq K \cap (x)$ and by Lemma 3.4, we have $(x) \subseteq K$. Thus $S_i \subseteq K$. Let $r \in R$ and $x, y \in S_i$. Since $rx, x - y \in K$ we have either each of $(rx)$ and $(x - y)$ may be uniform or equal to $(0)$. This implies $rx, x - y \in X_i$. This shows that $X_i$ is an uniform submodule of $K$.

*Lemma* 3.6—(i) If $U$ is an uniform submodule of $R$ then $E(U) = E(X_i)$ for some $i \in I$. (ii) Each $E(X_i)$ is indecomposable. (iii) If $R$ has *FGD* and $n = \dim R$ then there exist $\mathscr{U} = \{E(X_{m_i})/m_i \in I \text{ and } 1 \leqslant i \leqslant n\}$ such that $j \in I$ implies $E(X_j) \cong E(X_{m_i})$ for some $1 \leqslant i \leqslant n$. (iv) The number of the non-isomorphic indecomposable injective modules in $\mathscr{U}$ is $\leqslant n$.

Proof : (i) If $0 \neq a \in U$ then $a \in S_j$ for some $j \in I$ and hence $E(X_j) = E((a)) = E(U)$. (ii) follows from the fact that each $X_i$ is uniform. (iii) By Lemma 3.2, there exist indecomposable injective modules $K_i$, $1 \leqslant i \leqslant n$ such that $E(R) = \bigoplus_{i=1}^{n} K_i$ and $K_i = E(U_i)$ for some uniform submodule $U_i$ of $R$. Thus $K_i = E(U_i) = E(X_j)$ for some $j \in I$. Thus $\{K_i/1 \leqslant i \leqslant n\} \subseteq \{E(X_j)/j \in I\}$ and for each $1 \leqslant i \leqslant n$ there exists $m_i \in I$ such that $K_i = E(X_{m_i})$. Let $j \in I$ and consider $X_j$. By Lemma 3.5, $X_j$ is uniform and following the proof of the second part of Lemma 1.9 (*a*) of Chattars and Hajarnivas[2], we can find uniform submodules $A_i$, $2 \leqslant i \leqslant n$ such that the sum $X_j + A_2 + \ldots + A_n$ is direct and essential in $R$. Following the steps of the proof of Lemma 3.2, we get $E(R) = E(X_j) \oplus E(A_2) \oplus \ldots \oplus E(A_n)$, and so by Corollary (p 69 of Sharpe and Vamaos[7]) we have that $E(X_j) \cong K_i$ for some $1 \leqslant i \leqslant n$. Hence $E(X_j) \cong E(X_{m_i})$. (iv) follows from (iii).

*Lemma* 3.7—If $R$ has FGD, $U$ is an uniform module and $0 \neq a \in U$ such that $(0 : a)$ is a maximal complement submodule of $R$ then $E(U) \cong E(X_i)$ for some $E(X_i) \in \mathscr{U}$ (cf. Lemma 3.6 iii).

Proof : By Result 1.9, there exists an uniform submodule $L$ of $R$ such that $(0 : a) \cap L = (0)$ and $(0 : a) \oplus L \leqslant_e R$. Write $K = ((0 : a) \oplus L)/(0 : a)$. By the

main Theorem of Yenumula and Bhavanari[8], we have that $K \leqslant_e R/(0 : a)$. Since $K \cong L$ we have $K$ is also uniform and so $R/(0 : a)$ is uniform too. By Lemma 3.6, $E(L) \cong E(X_i)$ for some $E(X_i) \in \mathscr{U}$ and since $K \cong L$ we have $E(K) \cong E(L)$. Since $K \leqslant_e R/(0 : a)$ we have $E(R/(0 : a)) \cong E(L)$. Since $Ra \leqslant_e U$ and $Ra \cong R/(0 : a)$ we have $E(U) = E(Ra) \cong E(L) \cong E(X_i)$.

Write $\mathscr{L}$ = the set of all essential $E$-irreducible submodules of $R$. Define an equivalence relation '$\approx$' on $\mathscr{L}$ as : $S \approx L \Leftrightarrow E(R/S) \simeq E(R/L)$ for all $S, L \in \mathscr{L}$. Suppose the disjoint equivalence classes are $\{\mathscr{L}_i\}_{i \in J}$. For each $i \in J$, select an element $H_i \in \mathscr{L}_i$ and write $E_i = E(R/H_i)$. Since $H_i$ is essential and $E$-irreducible by Result 1.10, $H_i$ is irreducible and by Lemma 2.1 (i), $R/H_i$ is uniform. So each $E_i$ is indecomposable. Since $H_i$'s are selected from distinct equivalence classes, by the definition of the equivalence relation, the elements of $\mathscr{E} = \{E_i / i \in J\}$ are distinct non-isomorphic indecomposable injective modules.

*Lemma* 3.8—Let $U$ be an uniform module and $a \in U$ such that $(0 : a)$ is an essential and $E$-irreducible submodule of $R$. Then $E(U) \cong E_i$ for some $E_i \in \mathscr{E}$.

PROOF : Since $(0 : a) \in \mathscr{L}$ we have $(0 : a) \in \mathscr{L}_i$ for some $i \in J$ and so $E(R/(0 : a)) \cong E(R/H_i) = E_i$. Thus $E(U) = E(Ra) \cong E(R/(0 : a)) \cong E_i$.

Combining Lemma 3.7 and 3.8, we find :

*Corollary*—If there is an uniform module $U$ and $a, b \in U$ such that $(0 : a)$ is a maximal complement submodule and $(0 : b)$ is an essential $E$-irreducible submodule of $R$, then $E(X_i) \cong E_j$ for some $E(X_i) \cup \mathscr{U}$ and $E_j \in \mathscr{E}$.

*Notation* : We denote the set $\mathscr{U} \in \mathscr{E}$ by $\mathscr{H}$.

*Theorem* 3.3—Let $R$ be a ring with unity and $M$ an unitary $R$-module. If $R$ has FGD and $n = \dim M$ then $E(M) = \overset{n}{\underset{i=1}{\oplus}} K_i$ where each $K_i \cong L_i$ for some $L_i \in \mathscr{H}$.

PROOF : By Lemma 3.2, there exist indecomposable injective modules $K_i$, $1 \leqslant i \leqslant n$ such that $E(M) = \overset{n}{\underset{i=1}{\oplus}} K_i$. Fix $i$ $(1 \leqslant i \leqslant n)$. If there is an element $0 \neq a \in K_i$ such that $(0 : a)$ is a maximal complement in $R$, then by Lemma 3.7, we have $K_i \cong L_i$ for some $L_i \in \mathscr{U} \subseteq \mathscr{H}$. Otherwise, by Theorem 2.1, there exists $0 \neq b \in K_i$ such that $(0 : b)$ is an essential $E$-irreducible submodule of $R$. In this case by Lemma 3.8, $K_i \cong L$ for some $L \in \mathscr{E} \subseteq \mathscr{H}$. Thus for each $i$ $(1 \leqslant i \leqslant n)$ we have $K_i \cong L$ for some $L \in \mathscr{H}$.

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# ON COMPLETE INTEGRAL CLOSURE OF *G*-DOMAIN

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Let $R$ be a commutative ring with identity 1 and let $R'$ be the total quotient ring of $R$. It is well known that the complete integral closure $R_0$ of $R$ in $R'$ is a subring of $R'$. However $R_0$ is not necessarily itself completely integrally closed. We denote by $R^{(1)} = R_0$, $R^{(2)} = (R_0)_0 = R_{00}$ and for positive integer $n \geqslant 3$, $R^{(n)} = (R^{(n-1)})_0$. As pointed out by Professor Silvio Greco, it will be quite interesting to find classes of rings $R$ for which there exists an integer $n \geqslant 1$ such that $R^{(n)} = R^{(n+1)}$ and hence $R^{(n)} = R^{(m)}$ for all $m \geqslant n$. For Noetherian domain $R$, it is well known that $R^{(1)} = R^{(2)}$. In this paper we show that if $R$ is a $G$-domain then $R^{(2)} = R^{(3)}$, i.e. $R_{00} = R_{000}$ (Note that there is a $G$-domain $R$ for which $R^{(1)} \neq R^{(2)}$ i.e $R_0 \neq R_{00}$).

## 1. Almost Integral Dependence

Let $R$ be a commutative ring with identity 1. Let $R'$ be the total quotient ring of $R$. An element $a \in R'$ is called almost integral over $R$ if there exists a finitely generated submodule of the $R$-module $R'$ which contains all powers of $a$. The ring $R_0$ of elements of $R'$ which are almost integral over $R$ is called the complete integral closure of $R$ in $R'$. If $R_0 = R$, then $R$ is called completely integrally closed. Some important results on complete integral closure have been proved by Gilmer and Heinzer[2]. The following theorem is useful for determining the complete integral closure of a ring.

*Theorem* 1.1—(Larsen and McCarthy[6], Theorem 4.20)—The complete integral closure of a ring $R$ with total quotient ring $R'$ is

$$R_0 = \left[ \begin{array}{l} x : x \in R' \text{ and there exists a regular element } r \in R \\ \text{such that } rx^n \in R \text{ for } n = 0, 1, 2, \ldots \end{array} \right]$$

It is well known that $R_0$ is not necessarily completely integrally closed. (Larsen and McCarthy[6], Ch. IV, Ex. 14).

## 2. *G*-domains

An integral domain $R$ with quotient field $K$ is called a $G$-domain if it satisfies one of the following equivalent conditions

1. $K$ is a finitely generated $R$-algebra
2. $K$ can be generated by one element as an $R$-algebra.

These rings were first introduced and studied independently by Goldman[3] and Krull[5] at the same time. Later Noetherian $G$-domains were studied by Artin and Tate[1]. See also Kaplansky[4], 1.3.

It is shown (Singh[7]) that there is a $G$-domain $R$ such that $R_0$ is not completely integrally closed i. e. $R_0 \neq R_{00}$.

The following theorem shows that if $R$ is $G$-domain then $R_{00}$ is completely integrally closed.

*Theorem* 2.2—If $R$ is a $G$-domain, then $R_{000} = R_{00}$.

We prove this theorem in a series of lemmas.

*Lemma* 1—Let $R$ be a $G$-domain with quotient field $K$. If $x \in K$ be such that $x^k \in R_0$ for some positive integer $k$, then $x \in R_0$.

PROOF : Let $K = R[u^{-1}]$. There exist positive integers $t_1$ $t_2$, . , $t_{k-1}$ such that $x^i \in u^{-t_i} R$ for $i = 1, 2, \ldots, k-1$.

Put $t = \max\{t_1, t_2, \ldots, t_{k-1}\}$. Therefore $u^t x^i \in R$ for $i = 1, 2, \ldots, k-1$. Since $x^k \in R_0$, there exists a non-zero element $c$ in $R$ such that $c\,(x^k)^n \in R$ for $n = 0, 1, 2,$ .. Writing each positive integer $m$ as $m = kn + r$ where $0 \leqslant r \leqslant k-1$, we have $(u^t c)\ x^m = (u^t\, c).\ x^{kn+r} = (u^t\, x^r)\ c\,(x^k)^n \in R.\ R \subseteq R$. Therefore $x \in R_0$.

*Lemma* 2—Let $R$ be a $G$-domain with quotient field $K$ and let $K = R\,[u^{-1}]$. Let $N = \{x \in K : x^n \in uR$ for some positive integer $n\}$ and $T = \{x \in K : x\,N \subseteq N\}$. Then $T$ is a subring of $K$ containing $R$, and hence $R_{00} \subseteq T_{00}$.

PROOF : First we observe that if $R = K$, then $R = N = T = K$ and there is nothing to prove. Suppose $R \neq K$. In this case one can easily verify that ...

$u^2 R \underset{\neq}{\subset} uR \underset{\neq}{\subset} R \underset{\neq}{\subset} u^{-1} R \underset{\neq}{\subset} u^{-2} R \ldots$ and $K = \bigcup_{n=1}^{\infty} u^{-n} R$. We now prove that $N$ is an $R$-submodule of $K$. It is clear that $0 \in N$ and that $x \in N$, $a \in R$ implies $a\,x \in N$. Now suppose $x, y \in N$. We can find a positive integer $n$ such that $x^n \in u\,R$ and $y^n \in uR$. Choose a positive integer $k$ such that $x, x^2, \ldots x^{n-1}, y, y^2, \ldots, y^{n-1} \in u^{-k} R$. Now every positive integer $m$ can be written as $m = qn + r$ where $0 \leqslant r \leqslant n-1$. Therefore $x^m = (x^n)^q.\ x^r \in (uR)^q\, u^{-k}\, R \subseteq u^{\,k+1} R$. Similarly $y^m \in u^{-k+1} R$. Take $n_0 = 2kn$. For $j \geqslant n_0$, writing $j = tn_0 + s$ where $0 \leqslant s \leqslant n_0 - 1$, we have $x^j = (x^n)^{2kt}\, x^s \in (u^{2kt} R)\,(u^{-k+1} R) \subseteq u^k R$ and $y^j \in u^k R$. Therefore $(x+y)^{2n}.\ O = \sum_{i+j=2n_0} x^i y^j \in \Sigma$

$(u^{-k+1} R)\,(u^k R) \subseteq u\,R$ and hence $x + y \in N$.

Now since $N$ is an $R$-submodule of $K$, it is clear that $T$ is a subring of $K$ containing $R$.

*Lemma* 3—Let $R$ be a $G$-domain with quotient field $K$ and let $T$ be as defined in Lemma 2; Then $T \subseteq R_{00}$.

PROOF : Let $x \in T$. Note that $u \neq 0$, $u \in N$ By definition of $T$, we have $xu \in N$. and so there exists a positive integer $n$ such that $(xu)^n \subset uR \subseteq R \subseteq R_0$.

Using Lemma 1, we have $xu \in R_0$. Since $x \in T$ implies $x^n \in T$ for every positive integer $n$, by the same argument we get $ux^n \in R_0$ for $n = 0, 1, 2, \ldots$ and hence $x \in R_{00}$. Since $x$ was an arbitrary element of $T$, we get $T \subseteq R_{00}$.

*Lemma* 4—Let $R$ be a $G$-domain with quotient field $K$ and $T$ be subring of $K$ as defined in Lemma 2; then $T$ is completely integrally closed. In particular $T = T_{00}$.

PROOF : Let $x \in T_0$. There exists a non-zero element $c$ in $T$ such that $cx^n \in T$ for $n = 0, 1, 2, \ldots$. Since $K = R[u^{-1}]$ there exists a positive integer $k$ such that $c^{-1} \in u^{-k} R$. Choose an arbitrary element $y$ in $N$ and fix it. There exists a positive integer $t$ such that $y^t \in uR$; therefor $y^{t(k+1)} \in u^{k+1} R$ and hence $c^{-1} y^{t(k+1)} \in uR \subseteq N$; and since $cx^n \in T$, this implies $x^n y^{t(k+1)} \in N$ for $n = 0, 1, 2, \ldots$

Taking $n = t(k+1)$ we get that $(xy)^{t(k+1)} \in N$ and hence $xy \in N$. Since $y$ was an arbitrary element of $N$, we get that $x \in T$. Therefore $T_0 \subseteq T$, i. e $T$ is completely integrally closed.

*Proof of Theorem* 2.2—By Lemma 2 and Lemma 4, $R_{00} \subseteq T$. Using Lemma 3, we get that $R_{00} = T$. Now using Lemma 4 again, we get $R_{000} = R_{00}$, i.e $R_{00}$ is completely integrally closed.


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# ON ALMOST UNIFIED CONTACT FINSLER STRUCTURES AND CONNECTIONS

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The purpose of the present paper is to unify the almost contact Finsler structure[2] and almost para contact Finsler structure[1,3] on a differentiable manifold $M$, and to determine all Finsler connections compatible with the unified structure.

## 1. INTRODUCTION

Let $M$ be an $n$-dimensional $C^\infty$-differentiable manifold equipped with a Finsler connection $F\,\Gamma = (F, N, C)$ in the sense of Matsumoto[4]. Then the $h$- and $v$-covariant derivatives of a Finsler tensor field $K$ of type (1.1) is given below :

$$\left.\begin{aligned} K^i_{j|k} &\quad \delta K^i_j/\delta x^k + F^i_{mk} K^m_j - F^m_{jk} K^i_m \\ K^i_{j|k} &= \partial K^i_j/\partial y^k + C^i_{mk} K^m_j - C^m_{jk} K^i_m \end{aligned}\right\} \qquad \text{...(1.1)}$$

where

$$\delta/\delta x^k = \partial/\partial x^k - N^i_k\ (\partial/\partial y^i).$$

If $F\,\Gamma = (F, N, C)$ and $F\bar{\Gamma} = (\bar{F}, \bar{N}, \bar{C})$ are two Finsler connections on $M$, then the transformation from $F\,\Gamma$ to $F\bar{\Gamma}$ is given by:

$$\left.\begin{aligned} \bar{N}^i_j &= N^i_j - A^i_j \\ \bar{F}^i_{jk} &= F^i_{jk} + C^i_{jh}\ A^h_k - B^i_{jk} \\ \bar{C}^i_{jk} &= C^i_{jk} - D^i_{jk} \end{aligned}\right\} \qquad \text{...(1.2)}$$

where $A$, $B$, $D$ are the difference tensor fields[4].

## 2. ALMOST UNIFIED CONTACT FINSLER STRUCTURES

An almost unified contact Finsler structure on $M$ of odd dimension is defined by

a triad $(\phi, \eta, \xi)$ of a Finsler tensor field $\phi(x, y)$ of type (1,1), a $l$-form $\eta(x, y)$ and a vector field $\xi(x, y)$ satisfying the following conditions :

$$\left.\begin{aligned} &\phi_j^a \phi_a^k = \mu \left\{ \delta_j^k - \eta_j \xi^k \right\} ; \text{rank} \left( \phi_j^a \right) = n - 1 \\ &\phi_j^a \eta_a = 0 \\ &\phi_j^a \xi^j = 0 \\ &\eta_a (\xi^a) = 1 \end{aligned}\right\} \quad \ldots(2.1)$$

where $\mu$ is some non-zero constant.

*Remark* : When $\mu = +1$, the structure $(\phi, \eta, \xi)$ defined above becomes an almost para-contact structure on $M$ and when $\mu = -1$, it becomes an almost contact Finsler structure on $M$.

Analogous to Obata operators[6], we have,

$$\left.\begin{aligned} &\underset{1}{0} = 1/2 \left(I \otimes I + 1/\mu (\phi \otimes \phi)\right) \\ &\underset{2}{0} = 1/2 \left(I \otimes I - 1/\mu (\phi \otimes \phi)\right) \\ &\Omega = 1/2 \left(\eta \otimes \xi \otimes I + I \otimes \eta \otimes \xi - \eta \otimes \xi \otimes \eta \otimes \xi\right) \end{aligned}\right\} \quad \ldots(2\ 2)$$

where $I$ is the identity operator.

The operators act, for example, in the following way:

$$(\underset{1}{0}.\ t)^k_{mj} = \underset{1}{0}{}^{kh}_{ij}\ t^i_{hm} \quad (\underset{1}{0},\ \Omega)^{kb}_{mj} = \underset{1}{0}{}^{kh}_{ij}\ \Omega^{ib}_{mh} \quad \ldots(2.3)$$

where $t$ is an arbitrary Finsler tensoı field of type (1, 2). They satisfy the following relations.

$$\text{(a)}\quad \underset{1}{0} + \underset{2}{0} = I \otimes I ;\ \underset{2}{0}.\ \underset{1}{0} = \underset{1}{0}.\ \underset{2}{0} = \underset{2}{0}.\ \Omega = \Omega\ .\underset{2}{0}$$

$$= \underset{1}{0}\ \Omega = \Omega.\ \underset{1}{0} = \Omega.\ \Omega = \Omega/2$$

$$\text{(b)}\quad (\underset{2}{0} + \Omega)\ (\underset{1}{0} - \Omega) = (\underset{1}{0} - \Omega)\ (\underset{2}{0} + \Omega) = 0.0. \quad \ldots(2.4)$$

If we put $\underset{1}{Q} = \underset{1}{0} - \Omega$ and $\underset{2}{Q} = \underset{2}{0} + \Omega$, then (2.4b) becomes

$$\underset{2}{Q}.\ \underset{1}{Q} = \underset{1}{Q}.\ \underset{2}{Q} = 0.\ 0$$

*Lemma* 2.1—A system of tensor equations $\underset{2}{Q}\,X = A$ with $X$ as unknowns has solutions if and only if $\underset{1}{Q}\,A = 0$. Then the general solution of $\underset{2}{Q}.\,X = A$ is,

$$X = A + \underset{1}{Q} \cdot Y$$

where $Y$ is an arbitrary Finsler tensor field of the same type as that of $X$.

## 3. Finsler Connections Compatible with Almost Unified Contact Finsler Structure

*Definition* 3.1—A Finsler connection $F\Gamma = (F, N, C)$ is called an almost unified contact Finsler connection relative to $(\phi, \eta, \xi)$ if :

$$\left.\begin{aligned} &\phi^i_{j|k} = 0, \phi^i_{j|k} = 0, \eta_i{}^{|}{}_k = 0, \eta_{i|k} = 0 \\ &\xi^i_{|k} = 0, \xi^i_{|k} = 0. \end{aligned}\right] \qquad \ldots(3.1)$$

It can be shown that for any almost unified contact Finsler connection the operators $\underset{1}{0}$, $\underset{2}{0}$, $\underset{1}{Q}$, $\underset{2}{Q}$ and $\Omega$ are $h$- and $v$-covariant constants.

Giving a Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ on $M$, any Finsler connection $F\Gamma = (F, N, C)$ on $M$ can be expressed in terms of the difference tensors $X^i_j$ , $B^i_{jk}$ and $D^i_{jk}$ as :

$$\left.\begin{aligned} N^i_j &= \overset{0}{N}{}^i_j - X^i_j \\ F^i_{jk} &= \overset{0}{F}{}^i_{jk} + \overset{0}{C}{}^i_{jm} X^m_k - B^i_{jk} \\ C^i_{jk} &= \overset{0}{C}{}^i_{jk} - D^i_{jk}\,. \end{aligned}\right\} \qquad \ldots(3.2)$$

In order that $F\Gamma = (F, N, C)$ be an almost unified contact Finsler connection, we obtain by Lemma (2.1) as :

$$\begin{aligned} B^i_{jk} = 1/2 \Big\{ &- \xi^i \Big( \eta_j\, \eta_h\, \xi^{h\,0}_{\ \downarrow k} + \xi^{h\,0}_{\ |a}\, X^a_k + \eta_j \overset{0}{|}_k \\ &+ \eta_j \overset{0}{|}_a X^a_k \Big) + 2\eta_j \Big( \xi^{i0}_{\ \downarrow k} + \xi^{i0}_{\ |a} X^a_k \Big) \\ &+ 1/\mu\, \phi^h_j \Big( \phi^{i\,0}_{h\,|k} + \phi^{i\,0}_{h\,|a} X^a_k \Big) \Big\} + \underset{1}{Q}{}^{ih}_{aj}\, Y^a_{hk} \end{aligned}$$

$$D^{i}_{jk} = 1/2 \left\{ - \xi^{i} \left( \eta_{j} \eta_{h} \xi^{h0}_{\ \ |k} + \eta_{j} \overset{0}{|}_{k} \right) + 2 \eta_{i} \xi^{i0}_{\ \ |k} + 1/\mu\, \phi^{h}_{j} \phi^{i\,0}_{h\,|k} \right\} + \underset{1}{Q}^{ih}_{aj} Z^{a}_{hk}$$

where $Y$ and $Z$ are arbitrary Finsler tensor field of type (1,2) (cf. Miron and Hashiguchi[5]).

Consequently, we have,

*Theorem* 3.1—The general family of the almost unified contact Finsler connections $F\Gamma = (F, N\, C)$ relative to an almost unified contact Finsler structure $(\phi, \eta, \xi)$ is given by :

$$\text{(a)} \quad N^{i}_{j} = \overset{0}{N}{}^{i}_{j} - X^{i}_{j}$$

$$\text{(b)} \quad F^{i}_{jk} = \overset{0}{F}{}^{i}_{jk} + \overset{0}{C}{}^{i}_{ja} X^{a}_{k} - 1/2 \left\{ - \xi^{i} \left( \eta_{j} \eta_{h} \xi^{h0}_{\ \ \downarrow k} + \xi^{h0}_{\ \ |a} X^{a}_{k} + \eta_{j} \overset{0}{|}_{k} + \eta_{j} \overset{0}{|}_{a} X^{a}_{k} \right) + 2\eta_{j} \left( \xi^{i0}_{\ \ \downarrow k} + \xi^{i0}_{\ \ |a} X^{a}_{k} \right) + 1/\mu \phi^{h}_{j} \left( \phi^{i0}_{h\,|k} + \phi^{i\,0}_{h\,|a} X^{a}_{k} \right) \right\} - \underset{1}{Q}^{ih}_{aj} Y^{a}_{hk}$$

$$\text{(c)} \quad C^{i}_{jk} = \overset{0}{C}{}^{i}_{jk} - 1/2 \left\{ - \xi^{i} \left( \eta_{j} \eta_{h} \xi^{h0k}_{\ \ \ |} + \eta^{j0}_{\ \ |k} \right) + 2\eta_{j} \xi^{i0}_{\ \ |k} + 1/\mu\, \phi^{h}_{j} \phi^{i\,0}_{h\,|k} \right\} - \underset{1}{Q}^{jk}_{aj} Z^{a}_{hk}. \quad \text{...(3.3)}$$

Particularly by putting $X^{i}_{j} = 0$, $Y^{i}_{jk} = 0$ and $Z^{i}_{jk} = 0$ in (3.3), we have,

*Theorem* 3.2—If the initial Finsler connection is $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ then the following Finsler connection

$$\text{(a)} \quad \overset{k}{N}{}^{i}_{j} = \overset{0}{N}{}^{i}_{j}$$

$$\text{(b)} \quad \overset{k}{F}{}^{i}_{k} = \overset{0}{F}{}^{i}_{jk} - 1/2 \left\{ - \xi^{i} \left( \eta_{j} \eta_{h} \xi^{h0}_{\ \ |k} + \eta_{j} \overset{0}{|}_{k} \right) + 2\, \eta_{j} \xi^{i0}_{\ \ |k} + 1/\mu\, \phi^{h}_{j} \phi^{i\,0}_{h\,|k} \right\}$$

$$\text{(c)}\quad \overset{k}{C}{}^{i}_{jk} = \overset{0}{C}{}^{i}_{jk} - 1/2\Big\{ - \xi^{i}\left(\eta_j\, \eta_h\, \xi^{h0}{}_{|k} + \eta_j\, {}^{0}_{|k}\right) + 2\,\eta_j\, \xi^{i0}{}_{|k} + 1/\mu\, \phi^{h}_{j}\ \phi^{i\,0}_{h\,|k}\Big\} \qquad ...(3.4)$$

is an almost unified contact Finsler connection

*Definition* 3.2—The Finsler connection $F\overset{k}{\Gamma} = (\overset{k}{F}, \overset{k}{N}, \overset{k}{C})$ given by (3.4) is called the almost unified Kawaguchi connection derived from $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$.

Now, let $F\overset{m}{\Gamma} = (\overset{m}{F}, \overset{m}{N}, \overset{m}{C})$ be the Matsumoto connection then,

$$\overset{m}{N}{}^{i}_{j} = \overset{0}{N}{}^{i}_{jk},\ \overset{m}{F}{}^{i}_{jk} = \partial \overset{0}{N}{}^{i}_{k} / \partial y^{j}\ \overset{m}{C}{}^{i}_{jk} = \overset{0}{C}{}^{i}_{jk}\,.$$

Replacing the initial Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ by the Matsumoto connection in (3.4), we get an almost unified contact Finsler connection denoted by $F\overset{q}{\Gamma} = (\overset{q}{F}, \overset{q}{N}, \overset{q}{C})$ called an almost unified canonical contact Finsler connection.

*Theorm* 3.3—If we replace the initial Finsler connection $F\overset{0}{\Gamma} = (\overset{0}{F}, \overset{0}{N}, \overset{0}{C})$ by an almost unified contact Finsler connections $F\Gamma = (F, N, C)$ then the general family of the almost unified contact Finsler connection $F\bar{\Gamma} = (\bar{F}, \bar{N}, \bar{C})$ is given by :

$$\bar{N}^{i}_{j} = N^{i}_{j} = X^{i}_{j}$$

$$\bar{F}^{i}_{jk} = F^{i}_{jk} + C^{i}_{ja}\ X^{a}_{k} - \underset{1}{Q}{}^{ih}_{aj}\ Y^{a}_{hk}$$

$$\bar{C}^{i}_{jk} = C^{i}_{jk} - Q^{ih}_{aj}\ Z^{a}_{hk}\,. \qquad ...(3.5)$$

The Finsler connections having the non-linear connection $N$ common is denoted by $F\Gamma\,(N)$.

*Theorem* 3.4—The family of the almost unified contact Finsler connections $F\ \bar{\Gamma}\,(N)$ is given by :

$$\text{(a)}\quad \bar{N}^{i}_{j} = N^{i}_{j}$$

$$\text{(b)}\quad \bar{F}^{i}_{jk} = F^{i}_{jk} - \underset{1}{Q}{}^{ih}_{aj}\ Y^{a}_{hk}$$

$$\text{(c)}\quad \bar{C}^{i}_{jk} = C^{i}_{jk} - \underset{1}{Q}{}^{ih}_{aj}\ Z^{a}_{hk} \qquad ...(3.6)$$

where $Y$ and $Z$ are arbitrary Finsler tensor fields of type (1, 2).

We notice that (3.6) determine the transformation $F(N) \rightarrow F\bar{\Gamma}(N)$ of the almost unified contact Finsler connections having a common non-linear connection $N$.

*Theorem* 3.5—The set of all transformations of the almost unified contact Finsler connections given by (3.6) is an abelian group with respect to the product of mapping and which is isomorphic to the additive group of the pairs of the Finsler tensor fields $(\underset{1}{Q}.\,Y;\ \underset{1}{Q}.\,Z)$.

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# ON *F*-ABSOLUTELY TRANSLATIVE SUMMABILITY METHODS

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We call the matrix $A = (a_{nk})$ $F$-regular if it transforms $F$ into $F$ and $f$-lim $A(x) = f$-lim $x$ for all $x$ belonging to $F$ where $F$ denotes the set of all almost convergent sequences. We write $A \sim A'$ to denote that the $F$-regular matrices $A$ and $A'$ are $F$-absolutely equivalent for all bounded sequences. In the present paper we show that $AB \sim A'B'$ if $A, A', B, B'$ are $F$-regular and if $A \sim A'$ and $B \sim B$ (where $AB$ denotes the composition product). We also show that an $F$-regular matrix is always $F$-absolutely translative for all bounded sequences.

## 1. INTRODUCTION

Let $l_\infty$ and $c$ be the Banach spaces of bounded and convergent sequences with the usual supremum norm.

Let $A = (a_{nk})$ be an infinite matrix and $x = (x_k)$ be sequence with complex terms. if the sequence

$$(A_n(x)) = (\sum_{k=0}^{\infty} a_{nk}\, x_k)$$

exists (i. e. the series on the right hand side converges for each $n$) then the sequence $A(x) = (A_n(x))$ is called the $A$-transform of $x = (x_k)$. The matrix $A = (a_{nk})$ is said to be regular if the $A$-transform of $x$ is convergent to the limit of $x$ for each $x \in c$. The regularity conditions of $A$ are well known[1].

It is shown in Lorentz[3] that a sequence $x = (x_n) \in l_\infty$ is almost convegent to $s$ if and only if

$$\lim_q \frac{1}{q+1} \sum_{i=0}^{q} x_{n+i} = s, \text{ uniformly in } n. \qquad \ldots(1)$$

We write $f$-lim $x = s$ whenever (1) holds. By $F$ we denote the linear space of all almost convergent sequences. It is called that a sequence $x$ is almost $A$-summable to $L$ if $f$-lim $A(x) = L$.

We call the matrix $A = (a_{nk})$ is $F$-reqular if it transforms $F$ into $F$ and $f$-lim $A(x) = f$-lim $x$ for each $x \in F$. The following theorem gives the necessary and sufficient conditions for a matrix to be $F$-regular.

*Theorem* 1.1—A matrix $A = (a_{nk})$ is $F$-regular if and only if

(i) $\sup_n \Sigma_k \mid a_{nk} \mid < \infty$

(ii) $f$-lim $a_{nk} = 0$ for each $k$

(iii) $f$-lim $\Sigma_k \ a_{nk} = 1$

$$\text{(iv)} \quad \lim \sum_{k=0}^{\infty} \frac{1}{q+1} \mid \sum_{i=0}^{q} (a_{n+i,k} - a_{n+i,k+1}) \mid = 0$$

uniformly in $n$ (Duran[2]).

Throughout the paper, the sums will be taken from $k = 0$ to $k = \infty$.

## 2. F-Absolute Equivalence of Summability Methods

Concerning with the $F$-absolute equivalence of $F$-regular summability methods we have the following definition :

*Definition* 2.1—Let $A = (a_{nk})$ and $B = (b_{nk})$ be two $F$-regular matrices and $x = (x_k)$ be a sequence for which

$$z'_n = \Sigma_k \, a_{nk} \, x_k \ \text{ and } z''_n = \Sigma_k \, b_{nk} \, x_k$$

exist.

Then $A$ and $B$ are said to be $F$-absolutely equivalent for a given class of sequences $(x_k)$ if

$$f\text{-lim} \left( z'_n - z''_n \right) = 0$$

i.e. either $\left( z'_n \right)$ and $\left( z''_n \right)$ both almost converge to the same value, or else neither of them almost converges but their difference almost converges to zero[4].

The following theorem is known.

*Theorem* 2.2—Let $A$ and $B$ be two $F$-regular matrices. Then $A$ and $B$ are $F$-absolutely equivalent for all bounded sequences if and only if

$$\lim \sum_{k=0}^{\infty} \frac{1}{1+q} \left| \sum_{i=0}^{q} (a_{n+i,k} - b_{n+i,k}) \right| = 0$$

uniformly in $n$ (Orhan[4]).

### 3. $F$-Absolute Equivalence of Composition Products $(AB)$ and $(BA)$

The composition products $(AB)$ and $(BA)$ are respectively given by

$$v_n = \sum_{p=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk}\, b_{kp} \right) z_p = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} a_{nk}\, b_{kp}\, z_p \qquad \ldots(2)$$

and

$$v'_n = \sum_{p=0}^{\infty} \left( \sum_{k=0}^{\infty} b_{nk}\, a_{kp} \right) z_p = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} b_{nk}\, a_{kp}\, z_p. \qquad \ldots(3)$$

If $A$, $B$ are $F$-regular and $(z_p)$ is bounded then, by Theorem 1.1, the double sums in (2) is absolutely convergent, so that inversion in the order of summation is justified and therefore

$$\sum_{p=0}^{\infty} z_p \sum_{k=0}^{\infty} a_{nk}\, b_{kp} = \sum_{k=0}^{\infty} a_{nk} \sum_{p=0}^{\infty} b_{kp}\, z_p$$

for all $n$; in other words

$$(A\,B)\,z = A\,(Bz). \qquad \ldots(4)$$

That is to say the result of transforming the sequence $z$ by the product $AB$ is the same as transforming $z$ by the matrix $B$ and then transforming the result by the matrix $A$.

It is known that $(AB)$ and $(BA)$ are absolutely equivalent for all bounded sequences if the $T$-matrices $A$ and $B$ are absolutely equivalent for all bounded sequences (Cooke[1], p. 133)

We may ask whether $(AB)$ and $(BA)$ are $F$-absolutely equivalent for all bounded sequences provided that the $F$-regular matrices $A$ and $B$ are $F$-absolutely equivalent for all bounded sequence. In fact we have more than we need (see Theorem 3.1 below)

In what follows we write

$$A \sim A'$$

to denote that the $F$-regular matrices $A$ and $A'$ are $F$-absolutely equivalent for all bounded sequences.

*Theorem* 3.1—If $A$, $A'$, $B$, $B'$ are $F$-regular matrices and if

$$A \sim A' \text{ and } B \sim B'$$

then

$$AB \sim A'B'.$$

PROOF : It is enough to prove that if $A$, $A'$, $C$ are $F$-regular and if $A \sim A'$, then

$$AC \sim A'C \tag{5}$$

and

$$CA \sim CA'. \qquad \text{...(6)}$$

For applying (5) with $C = B$ we get $AB \sim A'B$, applying (6) with $A$, $A'$ replaced by $B$, $B'$ and $C$ replaced by $A'$, we get $A'B \sim A'B'$, and the result follows.

To prove (5), let $z$ be any bounded sequence. Since $C$ satisfies (i) of Theorem 1.1, $Cz$ is also bounded. Hence, by the definition of assertion that $A \sim A'$, it follows that

$$f\text{-lim}\ \{(A\ (Cz))_n - (A'\ (Cz))_n\} = 0.$$

By (4), this gives that

$$f\text{-lim}\ \{((AC)\ z)_n - ((A'\ C)\ z)_n\} = 0$$

and, since this holds for any bounded $z$, (5) holds.

For (6), let $z$ be any bounded sequence. Then (again by definition)

$$f\text{-lim}\ \{(Az)_n - (A'\ z)_n\} = 0.$$

Since $C$ is $F$-regular, it follows that

$$f\text{-lim}\ \{[C\ (Az - A'\ z)_n]\} = 0.$$

Using (4), we get that

$$f\text{-lim}\ \{[(CA)\ z]_n - [(AA')\ z]_n\} = 0$$

and the result follows.

## 4. $F$-Absolute Translative Matrices

Let $z = (z_k)$ be a sequence. We define the sequence $w = (w_k)$ by $w_k = z_{k-1}$, (where we take $z_{-1} = 0$). As known, the convergence of $(z_k)$ implies the convergence of $(w_k)$ (to the same value) and conversely. A similar result also holds for almost convergent sequences. We may expect that, if $z$ is $A$-summable to $s$ then $w$ is $A$-summable

to $s$ and conversely. But this is not necessarily the case (see Cooke[1], p.p. 113-19 and Powell[5], p. 42). To discuss the corresponding problem for the almost summaility we first need :

*Definition* 4.1—Let $A$ be an $F$-regular matrix. We say that $A$ is $F$-absolutely translative for bounded sequences if, for all bounded $(z_n)$, $(z_n - z_{n-1})$ is almost $A$-summable to zero; that is to say that $(u_n)$ is almost convergent to zero, where

$$u_n = \sum_{k=0}^{\infty} a_{nk} (z_k - z_{k-1}). \qquad ...(7)$$

While this is the natural definition, we can deduce an alternative from which is more convenient for applications. We have

$$\sum_{k=0}^{m} a_{nk} (z_k - z_{k-1}) = \sum_{k=0}^{m-1} (a_{nk} - a_{n,k+1}) z_k + a_{nm} z_m. \qquad ...(8)$$

If $A$ is $F$-regular then it follows from (i) of Theorem 1.1 that, for fixed $n$, $a_{nm} \to 0$ as $m \to \infty$. Hence if $(z_m)$ is bounded, then $a_{nm} z_m \to 0$. Thus letting $m \to \infty$ in (8), we see that (7) may be written as

$$u_n = \sum_{k=0}^{\infty} (a_{nk} - a_{n,k+1}) z_k \qquad ...(9)$$

Our final result is an analogue of Theorem 5.6. iv in Cooke[1] (p. 119).

*Theorem* 4.2—Let $A$ be an $F$-regular matrix. Then $A$ is always $F$-absolutely translative for all bounded sequences $(z_k)$.

PROOF : If $|z_k| \leqslant M$ (for all $k$), then

$$\left| \frac{1}{q+1} \sum_{t=0}^{q} u_{n+t} \right| = \left| \sum_{k=0}^{\infty} \frac{1}{q+1} \sum_{t=0}^{q} (a_{n+t,k} - a_{n+t,k+1}) z_k \right|$$

$$\leqslant M \sum_{k=0}^{\infty} \frac{1}{q+1} \left| \sum_{t=0}^{q} (a_{n+t,k} - a_{n+t,k+1}) \right| \to 0$$

as $q \to \infty$ uniformly in $n$, by (iv) of Theorem 1.1, whence the result.

Finally the authours are grateful to the referee for his kind remarks and suggestions which improved the presentation of the paper.

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# GROWTH OF COMPOSITE INTEGRAL FUNCTIONS

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In the present paper we study some growth properties of log $T(r, fg)$ relative to $T(r, f)$ and $T(r, g)$ for integral functions $f(z)$ and $g(z)$.

## 1. INTRODUCTION AND DEFINITIONS

Let $f(z)$ and $g(z)$ be two integral functions. We suppose that $T(r, f)$, $M(r, f)$, $N(r, a, f)$, $\delta(a, f)$, $\delta(a, (z), f)$, $\log^+ x$ etc. bear their usual meanings in the Nevanlinna theory of meromorphic functions (cf. Hayman[2]). Clunie[1] (see also Singh[7]) studied the comparative growths of $T(r, fg)$ with $T(r, f)$ and $T(r, g)$; he showed for transcendental integral functions $f(z)$ and $g(z)$ that $\lim\limits_{r\to\infty} \dfrac{T(r, fg)}{T(r, f)} = \infty$ and $\lim\limits_{r\to\infty} \dfrac{T(r, fg)}{T(r, g)} = \infty$. Singh[7] proved some comparative growth properties of $\log T(r, fg)$ and $T(r, f)$; also he raised the question of investigating the comparative growth of $\log T(r, fg)$ and $T(r, g)$ which he was unable to solve. In the present paper we prove a few theorems on the comparative growths of $\log T(r, fg)$ with $T(r, f)$ and, as well as, with $T(r, g)$. Throughout the paper we denote by $f(z)$ and $g(z)$ two integral functions with orders (lower orders) $\rho_f\ (\lambda_f)$ and $\rho_g\ (\lambda_g)$ respectively.

*Definition* 1—The number $\bar{\lambda}_g$ is said to be the hyper lower order of $g(z)$ if and only if

$$\bar{\lambda}_g = \liminf_{r\to\infty} \frac{\log\log\log M(r, g)}{\log r}.$$

It is clear that $\bar{\lambda}_g \leqslant \lambda_g$.

*Definition*—2[6]—A function $\rho_g(r)$ is called a proximate order of $g(z)$ relative to $T(r, g)$ if and only if (i) $\rho_g(r)$ is real, continuous and piecewise differentiable for $r > r_0$, (ii) $\lim\limits_{r\to\infty} \rho_g(r) = \rho_g$, (iii) $\lim\limits_{r\to\infty} r \log r\, \rho'_g(r) = 0$, (iv) $\limsup\limits_{r\to\infty} \dfrac{T(r, g)}{r^{\rho_g(r)}} = 1$.

*Proposition* 1—For $\delta > 0$ the function $r^{\rho_g + \delta - \rho_g(r)}$ is ultimately an increasing function of $r$.

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For,

$$\frac{d}{dr} r^{\rho_g + \delta - \rho_g(r)} = \{\rho_g + \delta - \rho_g(r) - r \log r \;\rho'_g(r)\}$$

$$r^{\rho_g + \delta - 1 - \rho_g(r)} > 0$$

for all sufficiently large values of $r$.

## 2. Theorems and Lemmas

Singh[7] proved a theorem on the estimation of $\limsup_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)}$, which after modification by Zhou[8] takes the following form.

*Theorem* 1—Let $f(z)$ and $g(z)$ be integral functions of finite orders such that $g(0) = 0$ and $\rho_g < \lambda_f \leqslant \rho_f$. Then $\lim_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = 0$.

Here we remark that for the truth of the above theorem the hypothesis $g(0) = 0$ is not essential. In the following we prove a comparative growth property of $\log T(r, fg)$ and $T(r, f)$ under some weaker hypotheses.

*Theorem* 2—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that $\lambda_g < \lambda_f \leqslant \rho_f < \infty$. Then $\liminf_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = 0$.

Proof : To prove the theorem we need the following lemma.

*Lemma* 1 (*Theorem* 1, *Niino and Suita*[4])—Let $f(z)$ and $g(z)$ be integral functions. If $M(r, g) > \dfrac{2+\epsilon}{\epsilon} \mid g(0) \mid$ for any $\epsilon > 0$, then we have

$$T(r, fg) < (1 + \epsilon)\, T(M(r, g), f).$$

In particular if $g(0) = 0$, then $T(r, fg) \leqslant T(M(r, g), f)$ for all $r > 0$.

*Proof of the Theorem*—In the present case for $\epsilon = 1$ and for all large values of $r$ we see that $M(r, g) > \dfrac{2+1}{1} \mid g(0) \mid$. So we obtain from Lemma 1 that for all large values of $r$

$$T(r, fg) \leqslant 2T(M(r, g), f). \qquad \text{...(1)}$$

Since $\lambda_g < \lambda_f$, we can choose $\epsilon\ (> 0)$ such that $\lambda_g + \epsilon < \lambda_f - \epsilon$. Also for all large values of $r$, $r^{\lambda_f - \epsilon/2} < T(r, f) < r^{\rho_f + \epsilon}$ and for a sequence of values of $r$ tending to infinity $\log M(r, g) < r^{\lambda_g + \epsilon}$.

Now from (1) we get for all large values of $r$

$$T(r, fg) \leqslant 2T(M(r, g), f) < 2\{M(r, g)\}^{\rho_f + \epsilon}$$

and so for all large values of $r$

$$\log T(r, fg) < \log 2 + (\rho_f + \epsilon) \log M(r, g).$$

Now for a sequence of values of $r$ tending to infinity we get

$$\log T(r, fg) < \log 2 + (\rho_f + \epsilon) r^{\lambda_g + \epsilon}$$

$$< \log 2 + (\rho_f + \epsilon) r^{\lambda_f - \epsilon}.$$

So for a sequence of values of $r$ tending to infinity we obtain

$$\frac{\log T(r, fg)}{T(r, f)} < \frac{\log 2}{r^{\lambda_f - \epsilon/2}} + \frac{\rho_f + \epsilon}{r^{\epsilon/2}} \text{ and hence } \liminf_{r \to \infty} \frac{\log T(r, fg)}{T(r, f)} = 0.$$

This proves the theorem.

Singh[7] proved the following theorem.

*Theorem* 3—Let $f(z)$ and $g(z)$ be integral functions of finite orders with $\rho_g > \rho_f$. Then $\limsup_{r \to \infty} \frac{\log T(r, fg)}{T(r, f)} = \infty$.

The analysis of the proof of Theorem 3 shows that the theorem is true, in general, only if $\lambda_f > 0$, which assumption is not explicitly stated in the theorem. The following example also strengthens this comment.

*Example* 1—Let $f(z) = z$ and $g(z) = e^z$. Then $\rho_f = \lambda_f = 0$ and $\rho_g = 1$, so $\rho_f < \rho_g$. Also $fg(z) = e^z$ and hence $\log T(r, fg) = \log r + O(1)$, $T(r, f) = \log r$, for $r > 1$. Therefore $\limsup_{r \to \infty} \frac{\log T(r, fg)}{T(r, f)} = 1$ which is contrary to Theorem 3.

In the following theorem we see that the conclusion of Theorem 3 can also be drawn even under somewhat relaxed hypotheses.

*Theorem* 4—Let $f(z)$ and $g(z)$ be two integral functions such that

$$0 < \lambda_f < \lambda_g < \infty. \text{ Then } \limsup_{r \to \infty} \frac{\log T(r, fg)}{T(r, f)} = \infty.$$

PROOF : We know that for $r > 0$ (Niino and Yang[5])

$$T(r, fg) \geq \frac{1}{3} \log M\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f\right\}. \quad \ldots(2)$$

Since $\lambda_f$ and $\lambda_g$ are the lower orders of $f(z)$ and $g(z)$ respectively, for given $\epsilon$ ($0 < \epsilon < \lambda_f$) and for all large values of $r$ we get $\log M(r, f) > r^{\lambda_f - \epsilon}$ and $\log M(r, g)$

$> r^{\lambda_g - \epsilon}$. So from (2) we get for all large values of $r$

$$T(r, fg) \geqslant \tfrac{1}{3}\{\tfrac{1}{8}\ M(r/4, g) + o(1)\}^{\lambda_f - \epsilon}$$

$$\geqslant \tfrac{1}{3}\{\tfrac{1}{9}\ M\ (r/4, g)\}^{\lambda_f - \epsilon}$$

which gives for all large values of $r$

$$\log T(r, f g) \geqslant O(1) + (\lambda_f - \epsilon) \log\ M\ (r/4, g)$$

$$\geqslant O\ (1) + (\lambda_f - \epsilon)\ (r/4)^{\lambda_g - \epsilon}. \qquad \ldots(3)$$

Also since $\liminf\limits_{r\to\infty} \dfrac{\log\ T(r, f)}{\log r} = \lambda_f$, it follows that for a sequence of values of $r$ tending to infinity $T(r, f) < r^{\lambda_f + \epsilon}$. Hence for a sequence of values of $r$ tending to infinity we obtain from (3) that

$$\frac{\log\ T(r, f\,g)}{T(r, f)} > \frac{O\ (1)}{r^{\lambda_f + \epsilon}} + (\lambda_f - \epsilon)\ (r/4)^{\lambda_g - \epsilon} \frac{1}{r^{\lambda_f + \epsilon}}$$

which gives $\limsup\limits_{r\to\infty} \dfrac{\log\ T(r, fg)}{T(r, f)} = \infty$ because we can choose $\epsilon$ $(0 < \epsilon < \lambda_f)$ such that $\lambda_f + \epsilon < \lambda_g - \epsilon$. This proves the theorem.

Now the following three theorems give estimations of the growth of the ratio $\dfrac{\log\ T(r, f\,g)}{T(r, g)}$, under different circumstances, as $r$ tends to infinity.

*Theorem* 5—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that $\rho_f$ and $\rho_g$ are finite. Then

$$\liminf_{r\to\infty} \frac{\log\ T(r, f\,g)}{T(r, g)} \leqslant 3.\ \rho_f . 2^{\rho_g}.$$

PROOF : It is well known that Hayman[2], p. 18.

$$T(r, f) \leqslant \log^{+} M(r, f) \leqslant 3\ T(2r, f) \qquad \ldots(4)$$

where $r > 0$ and $f(z)$ is an integral function. Also we know for integral functions $f(z)$ and $g(z)$ that for $r > 0$ (cf. Niino and Suita[4])

$$\log\ M(r, f\,g) \leqslant \log M\ (M(r, g), f). \qquad \ldots(5)$$

Since $f(z)$ and $g(z)$ are nonconstant and $\rho_f$ is the order of $f(z)$, we get for all large $r$ and given $\epsilon$ $(> 0)$ that

$$T(r, fg) < \log\ M\ (M(r, g), f) \leqslant \{M\ (r, g)\}^{\rho_f + \epsilon}.$$

So for all large $r$

$$\log T(r, fg) \leqslant (\rho_f + \epsilon) \log M(r, g) \qquad \ldots(6)$$

and hence

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} \leqslant (\rho_f + \epsilon) \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}.$$

Since $\epsilon$ ($> 0$) is arbitraty, it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r. g)} \leqslant \rho_f \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \qquad \ldots(7)$$

Let $\rho_g(r)$ be a proximate order of $g(z)$ relative to $T(r, g)$. Since $\limsup_{r\to\infty} \frac{T(r, g)}{r^{\rho_g(r)}} = 1$, it follows that for all large values of $r$ and for given $\epsilon$ $(0 < \epsilon < 1)$ $T(r,g) < (1 + \epsilon)\, r^{\rho_g(r)}$. From (4) we get, on replacement of $f$ by $g$, for all large values of $r$, $\log M(r, g) \leqslant 3T(2r, g) < 3(1 + \epsilon)(2r)^{\rho_g(2r)}$ and so for all large values of $r$

$$\log M(r, g) < 3(1 + \epsilon) \frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}}, \text{ where } \delta\ (> 0) \text{ is arbitrary.}$$

Since $r^{\rho_g+\delta-\rho_g(r)}$ is ultimately an increasing function of $r$, it follows that for all large $r$

$$\log M(r, g) < 3(1 + \epsilon) 2^{(\rho_g+\delta)}\, \sigma r^{\rho_g(r)}. \qquad \ldots(8)$$

Again since $\limsup_{r\to\infty} \frac{T(r, g)}{r^{\rho_g(r)}} = 1$, for a sequence of values of $r$ tending to infinity we abtain

$$T(r, g) > (1 - \epsilon)\, r^{\rho_g(r)}. \qquad \ldots(9)$$

From (8) and (9) we get for a sequence of values of $r$ tending to infinity

$$\log M(r, g) < 3\, \frac{1 + \epsilon}{1 - \epsilon}\, 2^{\rho_g+\delta}\, T(r, g)$$

which gives $\liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \leqslant 3\, \frac{1 + \epsilon}{1 - \epsilon}\, 2^{\rho_g+\delta}$. Since $\delta$ ($> 0$) and $\epsilon$ $(0 < \epsilon < 1)$ are arbitary, it follows that

$$\liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \leqslant 3.2^{\rho_g}. \qquad \ldots(10)$$

Theorem follows from (7) and (10). This proves the theorem.

*Theorem* 6—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that $\rho_f$ and $\lambda_g$ are finite. Also suppose that there exist integral functions $a_i(z)$) ($i = 1, 2, \ldots, n$; $n \leqslant \infty$) such that (i) $T(r, a_i(z)) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, \ldots, n$ and (ii) $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} \leqslant \pi.\, \rho_f.$$

Proof : To prove the theorem we require the following lemma.

*Lemma* 2[3]—Let $g(z)$ be an integral function with $\lambda_g < \infty$, and assume that $a_i(z)$ ($i = 1, 2, \ldots, n$; $n \leqslant \infty$) are entire functions satisfying $T(r, a_i(z)) = o\{T(r, g)\}$ then if

$$\sum_{i=1}^{n} \delta(a_i(z), g) = 1 \text{ we have } \lim_{r\to\infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

*Proof of the Theorem*—From (6) we obtain for all large values of $r$ and for $\epsilon$ ($> 0$) arbitrary

$$\log T(r, f\,g) \leqslant (\rho_f + \epsilon) \log M(r, g).$$

Hence we get $\limsup_{r\to\infty} \dfrac{\log T(r, f\,g)}{T(r, g)} \leqslant (\rho_f + \epsilon) \limsup_{r\to\infty} \dfrac{\log M(r, g)}{T(r, g)}$

and since $\epsilon$ ($> 0$) is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} \leqslant \rho_f \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \qquad \ldots(11)$$

The theorem follows from (11) and Lemma 2. This proves the theorem.

*Note* 1 : When, in particular, $a_i(z)$'s are constants the assumption (i) of Theorem 6 is obvious and so it need not be stated explicitly.

*Theorem* 7—Let $f(z)$ and $g(z)$ be two transcendental integral functions such that

(i) $\rho_g < \infty$ and the hyperlower order of $g(z)$, $\bar{\lambda}_g$ is positive

(ii) $\lambda_f > 0$, and

(iii) $\delta(0, f) < 1$.

Then

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} = \infty.$$

Proof : To prove the theorem we require the following lemma.

*Lemma* 3 (*Theorem* 5, *Niino and Suita*[4])—Let $f(z)$ be a transcendental integral function, $g(z)$ a transcendental integral function of finite order, $\eta$ a constant satisfying $0 < \eta < 1$, and $\alpha$ a positive number. Then we have

$$T(r, fg) + O(1) \geq N(r, 0, fg) > \log \frac{1}{\eta}\left[\frac{N\{M((\eta r)^{1/(1+\alpha)}, g), O, f\}}{\log M((\eta r)^{1/(1+\alpha)}, g) - O(1)} - O(1)\right]$$

as $r \to \infty$ through all values.

*Proof of the theorem*—Since $\delta(0, f) < 1$, for given $\epsilon > 0$ there exists a sequence of values of $r$ tending to infinity for which $\frac{N(r, 0, f)}{T(r, f)} > 1 - \delta(0, f) - \epsilon > 0$. Hence from Lemma 3 we get for a sequence of values of $r$ tending to infinity

$$T(r, fg) + O(1) \geqslant \log \frac{1}{\eta} \times \frac{(1 - \delta(0, f) - \epsilon)\, T\{M((\eta r)^{1/(1+\alpha)}, g), f\} - \log M((\eta r)^{1/(1+\alpha)}, g)\, O(1)}{\log M((\eta r)^{1/(1+\alpha)}, g) - O(1)} \quad \ldots(12)$$

Since $g(z)$ is of finite order $\rho_g$ it follows for given $\epsilon > 0$ and for all large values of $r$, $\log M(r, g) < r^{\rho_g + \epsilon}$. So from (12) we get for a sequence of values of $r$ tending to infinity

$$T(r, fg) + O(1) \geqslant \log \frac{1}{\eta} \times \frac{(1 - \delta(0, f) - \epsilon)\, T\{M((\eta r)^{1/(1+\alpha)}, g), f\} - \log M((\eta r)^{1/(1+\alpha)}, g)\, o(1)}{(\eta r)^{(\rho_g + \epsilon)/(1+\alpha)}\{1 - o(1)\}}$$

So for a sequence of values of $r$ tending to infinity

$$\log T(r, fg) + O(1) \geq O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\} + \log\left[1 - \frac{\log M((\eta r)^{1/(1+\alpha)}, g)\, O(1)}{(1 - \delta(0, f) - \epsilon)\, T\{M((\eta r)^{1/(1+\alpha)}, g), f\}}\right] \ldots (13)$$

Since $f(z)$ is transcendental, $\lim\limits_{r\to\infty} \frac{T(r, f)}{\log r} = \infty$ and so for given positive number $N$, however large, and for all large values of $r$ $T(r, f) > N \log r$. Therefore, we obtain from (13) for a sequence of values of $r$ tending to infinity

$$\log T(r, fg) + O(1) \geq O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\} + \log\left[1 - \frac{\log M((\eta r)^{1/(1+\alpha)}, g)\, O(1)}{(1 - \delta(0, f) - \epsilon)\, N \log M((\eta r)^{1/(1+\alpha)}, g)}\right]$$

(*equation continued on p.* 906)

$$= O(1) - O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\}$$

$$+ \log\left[1 - \frac{O(1)}{(1 - \delta(0, f) - \epsilon)N}\right]$$

where $N$ is so large that

$$1 - \frac{O(1)}{(1 - \delta(0, f) - \epsilon)N} > 0.$$

Hence, for a sequence of values of $r$ tending to infinity

$$\log T(r, fg) \geqslant O(1) - O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\}. \quad \ldots(14)$$

Since $g(z)$ is of finite positive hyper lower order $\overline{\lambda_g}$, it follows for all large values of $r$ that

$$\frac{\log\log\log M(r, g)}{\log r} \tfrac{1}{2}\,\overline{\lambda}_g.$$

i. e.,

$$\log M(r, g) > \exp(r^{1/2\,\overline{\lambda_g}}). \quad \ldots(15)$$

Again since $f(z)$ is of positive lower order $\lambda_f$, we get for all large values of $r$ and for $0 < M < \lambda_f$

$$\log T(r, f) > M \log r. \quad \ldots(16)$$

From (14), (15) and (16) we obtain for a sequence of values of $r$ tending to infinity

$$\log T(r, fg) \geqslant O(1) - O(\log r) + Me\,(\eta r)^{\overline{\lambda}_g\,/2(1+\alpha)}$$

which gives for a sequence of values of $r$ tending to infinity

$$\frac{\log T(r, g)}{T(r, g)} > O(1) - \frac{O(\log r)}{T(r, g)} + MT(r,g)\; e^{(\eta r)^{\overline{\lambda}_g/2(1+\alpha)}}$$

$$\geqslant O(1) + M\,\frac{e^{(\eta r)^{\overline{\lambda}_g/2(1+\alpha)}}}{r^{\rho_g+1}}$$

because $g(z)$ is transcendental and

$$\limsup_{r\to\infty} \frac{\log T(r, g)}{\log r} = \rho_g.$$

This inequality gives

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} = \infty.$$

This proves the theorem.

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# ON $L^1$-CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS

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We introduce here new modified cosine and sine sums

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \sin kx$$

respectively and study their $L^1$-convergence. We also deduce results about $L^1$-convergence of cosine and sine series.

## 1. Introduction

Consider cosine and sine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \qquad \ldots(1.1)$$

$$\sum_{k=1}^{\infty} a_k \sin kx \qquad \ldots(1.2)$$

or together

$$\sum_{k=1}^{\infty} a_k \phi_k(x) \qquad \ldots(1.3)$$

where $\phi_k(x)$ is $\cos kx$ or $\sin kx$ respectively. Let the partial sum of (1.3) be denoted by $S_n(x)$ and $t(x) = \lim_{n\to\infty} S_n(x)$.

The following results are known :

*Theorem A*[1] (p. 202)— If $\{a_n\}$ is a quasi-convex null sequence, then (1.1) is a Fourier series of its pointwise limit.

*Theorem* $B^{2,9}$—If $\{a_n\}$ is a quasi-convex null sequence, then (1.2) is a Fourier series if and only if

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.$$

If $a_k = o(1)$, $k \to \infty$ and

$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left( \frac{a_k}{k} \right) \right| < \infty \qquad \ldots(1.4)$$

we say that (1·1), (1.2) or (1.3) belongs to the class $R$.

Kano[4] generalized Theorems A and B by establishing the following results.

*Theorem C*—If (1.3) belongs to the class $R$, it is a Fourier series or equivalently, it represents an integrable function.

*Theorem D*—If $\{a_k\}$ is bounded and quasi-convex, the condition $\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$ is equivalent to (1.4).

Concerning the $L^1$-convergence of (1.2), Kano and Uchiyama[5] proved the following result :

*Theorem E*—Let $\{a_n\}$ be a bounded, quasi-convex sequence of real numbers. Then (1.2) converges in $L$ if and only if $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$ and $|a_n| \log n \to 0$ as $n \to \infty$.

Rees and Stanojević[7] introduced a cosine sum as $\frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx$.

Garrett and Stanojević[3], Ram[6] and Singh and Sharma[8] studied the $L^1$-convergence of this cosine sum under different sets of conditions on the coefficients $a_n$.

We introduce here new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) k \sin kx.$$

The aim of this paper is to study $L^1$-convergence of $f_n(x)$ and $g_n(x)$ and to obtain, analogous of Theorem $E$ and of the following classical Young-Kolmogorov theorem.

*Theorem* $F^1$ (p. 204)– If $\{a_n\}$ is a quasi-convex null sequence, then for the convergence of $\Sigma\, a_n \cos nx$ in the metric space $L$ it is necessary and sufficient that $\lim\limits_{n\to\infty} a_n \log n = 0$.

In what follows, $t_n(x)$ will represent $f_n(x)$ or $g_n(x)$.

## 2. Results

We prove the following result :

*Theorem* 1—Let (1.3) be in the class $R$. Then

$$\lim_{n\to\infty} t_n(x) = t(x) \text{ for } x \in (0, \pi] \text{ and } t \in L(0, \pi] \quad \ldots(2.1)$$

$$\| t_n - t \| = o(1), \; n \to \infty. \quad \ldots(2.2)$$

Proof : We will consider only cosine sums as the proof for the sine sums follows the same line. We have

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx$$

$$\times \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta\left(\frac{a_n}{n}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \quad \ldots(2.3)$$

where $\tilde{D}_n(x)$ denotes the conjugate Dirichlet kernel. Since $\{a_k\}$ is null, $\lim\limits_{n\to\infty} t_n(x) = t(x)$ for $x \in (0, \pi]$. Theorem C now implies that $t \in L(0, \pi]$.

The relation (2.3) yields

$$t(x) - t_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x)$$

$$= \lim_{m\to\infty} \frac{d}{dx}\left( \sum_{k=n+1}^{m} \frac{a_k}{k} \sin kx \right) + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x).$$

Applying Abel's transformation twice, we have

$$t(x) - t_n(x) = \lim_{m\to\infty} \left[ \sum_{k=n+1}^{m-1} \Delta\left(\frac{a_k}{k}\right) \widetilde{D}'_k(x) + \frac{a_m}{m} \widetilde{D}'_m(x) \right.$$

$$\left. - \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right] + \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x)$$

$$= \lim_{m\to\infty} \left[ \sum_{k=n+1}^{m-2} (k+1)\, \Delta^2\left(\frac{a_k}{k}\right) \widetilde{K}'_k(x) \right.$$

$$+ m\, \Delta\left(\frac{a_{m-1}}{m-1}\right) \widetilde{K}'_{m-1}(x)$$

$$- (n+1)\, \Delta\left(\frac{a_{n+1}}{n+1}\right) \widetilde{K}'_n(x)$$

$$\left. + \frac{a_m}{m} \widetilde{D}'_m(x) \right]$$

$$= \sum_{k=n+1}^{\infty} (k+1)\, \Delta^2\left(\frac{a_k}{k}\right) \widetilde{K}'_k(x)$$

$$- (n+1)\, \Delta\left(\frac{a_{n+1}}{n+1}\right) \widetilde{K}'_n(x)$$

where $\widetilde{K}_k(x)$ denotes the conjugate Fejér kernel. Thus

$$\int_{-\pi}^{\pi} | t(x) - t_n(x) |\, dx < \sum_{k=n+1}^{\infty} (k+1) \left| \Delta^2\left(\frac{a_k}{k}\right) \right| \int_{\pi}^{\pi} | \widetilde{K}'_k(x) |\, dx$$

$$+ (n+1) \left| \Delta\left(\frac{a_{n+1}}{n+1}\right) \right| \int_{-\pi}^{\pi} | \widetilde{K}'_n(x) |\, dx.$$

But, by Zygmund's Theorem[1] (p. 458), we have

$$\int_{-\pi}^{\pi} \left| \widetilde{K}_k'(x) \right| dx = O(k).$$

Moreover,

$$\left| \Delta \left( \frac{a_{n+1}}{n+1} \right) \right| = \left| \sum_{k=n+1}^{\infty} \Delta^2 \left( \frac{a_k}{k} \right) \right|$$

$$\leqslant \sum_{k=n+1}^{\infty} \frac{k^2}{k^2} \left| \Delta^2 \left( \frac{a_k}{k} \right) \right|$$

$$\leqslant \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} k^2 \left| \Delta^2 \left( \frac{a_k}{k} \right) \right|$$

$$= o\left( \frac{1}{(n+1)^2} \right)$$

by the given hypothesis. Thus, it follows that

$$\int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx = O\left( \sum_{k=n+1}^{\infty} (k+1)^2 \left| \Delta^2 \left( \frac{a_k}{k} \right) \right| \right) + o(1)$$

$$= o(1) \qquad ..(2.4)$$

by (1.4). This proves (2.2) and the Theorem 1 is proved.

## 3. Deductions

(i) If (1.3) belongs to the class $R$, then $\| t - S_n \| = o(1)$ $(n \to \infty)$ if and only if $| a_{n+1} | \log n = o(1)$, $n \to \infty$.

Proof : We prove this result for cosine series only, the proof for sine series being similar. Using (2.5), we get

$$\int_{-\pi}^{\pi} | t(x) - S_n(x) | \, dx = \int_{\pi}^{\pi} | t(x) - t_n(x) + t_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx$$

$$+ \int_{-\pi}^{\pi} | t_n(x) - S_n(x) | \, dx$$

(equation continued on p. 913)

$$= \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx = \int_{-\pi}^{\pi} | t_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | t(x) - S_n(x) | \, dx + \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx.$$

Since

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx = 0$$

by our Theorem 1 and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

behaves like $| a_{n+1} | \log n$ by Zygmund's Theorem, cited above, for large $n$, the conclusion of the corollary (i) follows.

(ii) We deduce now Theorem E of Kano and Uchiyama as follows :

The condition $| a_{n+1} | \log n = o(1)$, $n \to \infty$ and Theorem D imply that the sine series (1.2) belongs to the class $R$. The 'if part' now follows from deduction (i). The proof of the 'only if part' is the same as given by Kano and Uchiyama[5].

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# ON $\alpha$-QUASI CONVEX FUNCTIONS

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We define a new class of analytic functions $Q(\alpha)$ of $\alpha$-Quasi convex functions as follows: Let $\alpha \geqslant 0$. $Q(\alpha)$ is the class of all functions $f(z)$ holomorphic in the open unit disk $U = \{z : |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\left| (1 - \alpha) \arg \frac{zf'(z)}{g(z)} + \alpha \arg \frac{[zf'(z)]'}{g'(z)} \right| < \pi/2, \; z \in U$$

for some $\alpha$-convex function (Mocanu sense) $g(z)$. $Q(0)$ is the class $C$ of close-to-convex functions of Kaplan[4] and $Q(1)$ is the class $Q$ of quasi-convex functions of Noor and Thomas[3]. Thus $Q(\alpha)$ unifies these two classes of functions. We prove in this paper that when $\alpha \geqslant 1$ all functions in $Q(\alpha)$ are close-to-convex (hence univalent) and the class $Q(\alpha)$ is invariant under certain integral operator.

## 1. INTRODUCTION

Let $M_\alpha$ denote the class of $\alpha$-convex functions introduced by Mocanu[5]. This class $M_\alpha$ unifies the classes $K$- of convex univalent functions and $S^*$ of starlike univalent functions. In fact $M_0 = S^*$ and $M_1 = K$. $M_\alpha$ gives a continuous passage from the class of convex univalent functions to the class of starlike univalent functions as $\alpha$ decreases from 1 to 0.

Let $P(\alpha)$ denote the class of all functions $f(z)$ holomorphic in $U$ with $f(0) = 0$, $f'(0) = 1$ and satisfying for $\alpha \geqslant 0$

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} d\theta > -\pi$$

where

$$0 < \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi, \; z = re^{i\theta}, \; r < 1.$$

This class was introduced and studied by Bharathi[1]. Also it was proved[1] that a function $f \in P(\alpha)$ if and only if $\exists$ a starlike univalent function $g(z)$ such that

$$\text{Re} \left\{ \frac{z^\alpha f'^\alpha(z) f^{1-\alpha}(z)}{g(z)} \right\} > 0 \qquad \text{for } z \in U. \qquad \ldots(1)$$

Noor and Thomas[3] introduced the class $Q$ of Quasi-convex functions as the class of all functions $f$ holomorphic in $U$ with $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\operatorname{Re}\left\{\frac{[zf'(z)]'}{g'(z)}\right\} > 0, \quad z \in U \qquad \ldots(2)$$

for some convex univalent function $g(z)$ with $g(0) = 0$, $g'(0) = 1$. Noor and Thomas[3] also established that

$$K \subset Q \subset C.$$

The purpose of this paper is to introduce a new class $Q(\alpha)$ of $\alpha$-quasi convex functions. This new class will provide a continuous passage from $Q$ to $C$ as $\alpha$ decreases from 1 to 0 in an analogous manner $M_\alpha$ giving a continuous passage from $K$ to $S^*$.

Throughout this paper wherever a non-integral power of analytic function is being used, it is assumed that a suitable analytic branch has to be considered.

## 2. Main Results

We now define the new class $Q(\alpha)$ of $\alpha$-Quasi convex functions.

*Definition* 2.1—Let $\alpha \geqslant 0$. We denote by $Q(\alpha)$ the class of all functions $f(z)$ holomorphic in $U$ with $f(0) = 0$, $f'(0) = 1$ and for some $g \in M_\alpha$ satisfying

$$\left|(1-\alpha)\arg\frac{zf'(z)}{g(z)} + \alpha\arg\frac{[zf'(z)]'}{g'(z)}\right| < \pi/2, \; z \in U. \qquad \ldots(3)$$

*Remark* : It is clear that $Q(0)$ and $Q(1)$ are respectively the classes of close-to-convex functions and quasi-convex functions. Thus we have a continuous passage from $Q$ to $C$ as $\alpha$ decreases from 1 to 0. We also show that $\alpha$-quasi-convex function is a close-to-convex function for $\alpha > 1$ and hence univalent in $U$.

We need the following theorem to establish our main results:

*Theorem A*—Let $\beta, \nu \in \mathbb{C}$, $h(z)$ be convex univalent in $U$ with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \nu) > 0$, $z \in U$ and let $q(z)$ be holomorphic in $U$ with $q(0) = 1$ and $q(z) \prec h(z)$ ($q(z)$ is sub-ordinate to $h(z)$ in $U$). If $p(z) = 1 + p_1 z + \ldots$ is holomorphic in $U$ then

$$p(z) + \frac{zf'(z)}{\beta q(z) + \nu} \prec h(z) \Rightarrow p(z) \prec h(z).$$

For proof of the result. See Padmanabhan and Parvatham[6].

*Theorem* 2.1—For $\alpha > 1$, $Q(\alpha) \subseteq Q(0) = C$.

Proof : Let $f \in Q(\alpha)$. Then $\exists$ a $g \in M_\alpha$ such that

$$\left|(1-\alpha)\arg\frac{zf'(z)}{g(z)} + \alpha\arg\frac{[zf'(z)]'}{g'(z)}\right| < \pi/2, \; z \in U$$

equivalently,

$$\operatorname{Re}\left\{\frac{[zf'(z)]^{1-\alpha}\,[zf'(z)]'^{\alpha}}{g^{1-\alpha}(z)\,g'^{\alpha}(z)}\right\} > 0,\ z \in U.$$

Thus

$$\frac{[zf'(z)]'\,[zf'(z)]^{(1-\alpha)/\alpha}}{g^{(1-\alpha)/\alpha}(z)\,g'(z)} = p^{1/\alpha}(z) \qquad \ldots(4)$$

where

$$|\arg p(z)| < \pi/2.$$

Now let

$$P(z) = \frac{zf'(z)}{g(z)}.$$

Then

$$\frac{[zf'(z)]'}{g'(z)} = P(z) + \frac{zP'(z)}{zg'(z)/g(z)}.$$

Substituting in (4) we get,

$$\left[P(z) + \frac{zP'(z)}{\dfrac{zg'(z)}{g(z)}}\right] P^{(1-\alpha)/\alpha}(z) = p^{1-\alpha}(\alpha).$$

Setting

$$P^{1/\alpha}(z) = p_1(z)$$

and

$$\frac{zg'(z)}{g(z)} = q(z)$$

we get

$$p_1(z) + \frac{zp_1'(z)}{q(z)} = p^{1/\alpha}(z).$$

Since $g \in M_\alpha \subseteq S^*$, $\operatorname{Re} q(z) > 0$ in $U$ and $|\arg p^{1/\alpha}(z)| < \pi/2\alpha$, for $\alpha \geqslant 1$. $p^{1/\alpha}(z)$ lies in-side the convex set $|\arg \xi| < \pi/2\alpha$. Thus application of Theorem A with $\beta = 1$, $\gamma = 0$ and $h(z) = [(1-z)/(1+z)]^\alpha$ gives $|\arg p_1(z)| < \pi/2\alpha$ or $|\arg P(z)| < \pi/2\alpha$ which implies $f \in Q(0) = C$.

*Remark* : Since every function $f$ in $Q(\alpha)$ is a close-to-convex function, $Q(\alpha)$ is a proper sub-class of the class $S$ of normalized univalent function whenever $\alpha \geqslant 1$.

*Theorem* 2.2—$Q(\alpha) \subseteq Q(\beta)$ for $\alpha \geqslant 1$ and $0 \leqslant \beta \leqslant \alpha$.

PROOF : The case $\beta = 0$ has been established in Theorem 2.1, hence we assume $\beta > 0$. Let $f \in Q(\alpha)$. Then $\exists$ a $g \in M_\alpha$ such that for $z \in U$.

$$\left|(1-\alpha)\arg p(z) + \alpha \arg\left(p(z) + \frac{zp'(z)}{q(z)}\right)\right| < \pi/2$$

where

$$p(z) = \frac{zf'(z)}{g(z)}$$

and

$$q(z) = \frac{zg'(z)}{g(z)} .$$

Now

$$\left| (1-\beta) \arg p(z) + \beta \arg\left( p(z) + \frac{zp'(z)}{g(z)} \right) \right|$$

$$\leqslant \frac{\beta}{\alpha}\left| (1-\alpha) \arg p(z) + \alpha \arg\left( p(z) + \frac{zp'(z)}{q(z)} \right) \right|$$

$$+ \left(1 - \frac{\beta}{\alpha}\right) |\arg p(z)| \leqslant \frac{\beta}{\alpha}\frac{\pi}{2} + \left(1 - \frac{\beta}{\alpha}\right)\frac{\pi}{2} = \pi/2.$$

Thus $f \in Q(\beta)$ under the stated conditions of the theorem.

*Theorem* 2.3—$f \in Q(\alpha)$ if and only if $zf' \in P(\alpha)$.

PROOF : Since $f \in Q(\alpha)$ we have

$$\text{Re}\left\{ \frac{z^{\alpha} (zf'(z))^{1-\alpha} [zf'(z)]^{\cdot\alpha}}{z^{\alpha} g^{1-\alpha}(z)\, g'^{\alpha}(z)} \right\} > 0$$

for $z \in U$, where $g \in M_{\alpha}$. Equivalently, $\exists\ \phi \in S^*$ such that

$$Z^{\alpha} g^{1-\alpha}(z)\, g'^{\alpha}(z) = \phi(z)$$

and thus

$$\text{Re}\left\{ \frac{z^{\alpha} [zf'(z)]^{1-\alpha} [zf'(z)]'^{\alpha}}{\phi(z)} \right\} > 0,\ z \in U.$$

From the representation theorem for the class $P(\alpha)$[1] we have $zf' \in P(\alpha)$. Converse follows immediately from the representation theorem for the class $P(\alpha)$.

*Remark* : When $\alpha = 1$, $Q(\alpha)$ and $P(\alpha)$ are noting but the classes $Q$ and $C$ respectively. Thus Theorem 1 of Noor and Thomas[3] is a particular case of Theorem 2.3 When $\alpha = 0$, $Q(0) = C$ and $P(0) = CS^*$ of Reade[7]. Thus Theorem 2 3 reduces to the well-known result namely $f \in C$ if and only if $zf' \in CS^*$.

*Theorem* 2.4—For $\alpha$ real, $\bigcap_{\alpha \geqslant 1} Q(\alpha) = K$.

PROOF : Let $f \in Q(\alpha)$, then $\exists$ a $g \in M_{\alpha}$ such that for $z \in U$

$$\left| \left( \frac{1}{\alpha} - 1 \right) \arg \frac{zf'(z)}{g(z)} + \arg \frac{[zf'(z)]'}{g'(z)} \right| < \frac{\pi}{2\pi} .$$

Allowing $\alpha$ to $\infty$ we get,

$$\left| \arg \frac{[zf'(z)]'}{g'(z)} - \arg \frac{zf'(z)}{g(z)} \right| = 0$$

that is

$$\left| \arg \frac{[zf'(z)]'}{zf'(z)} \quad \frac{g(z)}{g'(z)} \right| = 0.$$

Therefore there exists a positive real number $m$ such that

$$\frac{[zf'(z)]'}{f'(z)} \cdot \frac{g(z)}{zg'(z)} = m$$

which is the same as saying

$$1 + \frac{zf''(z)}{f'(z)} = m \frac{zg'(z)}{g(z)}.$$

Since $g \in M_\alpha \subseteq S^*$ and $m > 0$ we get $f$ is a convex function. Hence from (2.2) it follows that $\bigcap_{\alpha \geqslant 1} Q(\alpha) \subseteq K$. Conversely, if $f$ is a convex function then by taking $g(z) = f(z)$ we see that $f \in Q(\infty)$. Thus $\bigcap_{\alpha \geqslant 1} Q(\alpha) \supseteq K$. Hence Theorem 2.4 follows.

*Remark* 3 : By Theorems 2.3 and 2.4 we get

$$\bigcap_{\alpha \geqslant 1} P(\alpha) = S^*.$$

*Theorem* 2.5—Whenever $f \in Q(\alpha)$ and $\alpha \geqslant 1$, then $F(z)$ defined by

$$F'(z) = \left\{ \frac{2/\alpha}{z^{2/\alpha}} \int_0^z t^{(2-\alpha)/\alpha} f'^{1/\alpha}(t)\, dt \right\}^\alpha \qquad \ldots(5)$$

is also in $Q(\alpha)$.

Proof : Since $f \in Q(\alpha)$ $\exists$ a $g \in M_\alpha$ such that

$$\operatorname{Re} \left\{ \frac{z^\alpha [zf'(z)]^{1-\alpha} [zf'(z)]'^{\alpha}}{z^\alpha g^{1-\alpha} g'^{\alpha}(z)} \right\} > 0 \quad \text{for } z \in U.$$

Let $G(z)$ be defined as,

$$G(z) = \left\{ \frac{1/\alpha}{z^{1/\alpha}} \int_0^z t^{(1-\alpha)/\alpha} g^{1/\alpha}(t)\, dt \right\}^\alpha.$$

Then

$$z^{1/\alpha} G^{1/\alpha}(z) = \frac{1}{\alpha} \int_0^z t^{(1-\alpha)/\alpha} g^{1/\alpha}(t)\, dt.$$

This on differentiation, with respect to $z$ gives

$$\frac{1}{\alpha}\, z^{(1-\alpha)/\alpha}\, G^{1/\alpha}(z) + \frac{1}{\alpha}\, z^{1/\alpha}\, G^{(1-\alpha)/\alpha}(z)\, G'(z) = \frac{1}{\alpha}\, z^{(1-\alpha)/\alpha}\, g^{1/\alpha}(z).$$

Hence,

$$zG'(z)\, G^{(1-\alpha)/\alpha}(z) + G^{1/\alpha}(z) = g^{1/\alpha}(z)$$

again differentiating this with respect to $z$ we get

$$\left(\frac{1}{\alpha} - 1\right) zG'^2(z)\, G^{(1-2\alpha)/\alpha}(z) + G'(z)\, G^{(1-\alpha)/\alpha}(z) + zG^{(1-\alpha)/\alpha}(z)\, G''(z)$$

$$+ \frac{1}{\alpha} G^{(1-\alpha)/\alpha}(z)\, G'(z)$$

$$= \frac{1}{\alpha}\, g^{(1-\alpha)/\alpha}(z)\, g'(z).$$

That is

$$(1-\alpha)\, zG^{(1-2\alpha)/\alpha}(z)\, G'^2(z) + \alpha\, zG^{(1-\alpha)/\alpha}(z)\, G''(z)$$

$$+ (1+\alpha)\, G^{(1-\alpha)/\alpha}(z)\, G'(z)$$

$$= g^{(1-\alpha)/\alpha}(z)\, g'(z).$$

Therefore,

$$G^{(1-\alpha)/\alpha}(z)\, G'(z)\left[(1+\alpha) + \frac{\alpha z G''(z)}{G'(z)} + (1-\alpha)\frac{zG'(z)}{G(z)}\right]$$

$$= g^{(1-\alpha)/\alpha}(z)\, g'(z) \qquad \ldots(6)$$

putting

$$p(z) = (1-\alpha)\,\frac{zG'(z)}{G(z)} + \left(1 + \frac{zG''(z)}{G(z)}\right)$$

(6) becomes

$$G^{(1-\alpha)/\alpha}(z)\, G'(z)\,[p(z) + 1] = g^{(1-\alpha)/\alpha}(z)\, g'(z) \qquad \ldots(7)$$

taking logarithamic differentiation of (7) we get

$$p(z) + \frac{\alpha z p'(z)}{p(z)+1} = (1-\alpha)\,\frac{zg'(z)}{g(z)} + \alpha\left(1 + \frac{zg''(z)}{g'(z)}\right).$$

As $g \in M_\alpha$ we get

$$\mathrm{Re}\left(p(z) + \frac{\alpha z p'(z)}{p(z)+1}\right) > 0$$

equivalently

$$p(z) + \frac{\alpha z p'(z)}{p(z)+1} \prec \frac{1+z}{1-z}.$$

Using a result due to Eenignburg *et al.*[2] we get

$$p(z) \prec \frac{1+z}{1-z} \quad \text{or} \quad \operatorname{Re} p(z) > 0.$$

Thus $G \in M_\alpha$ whenever $g \in M_\alpha$.

Now let

$$P(z) = \frac{[zF'(z)]^{1-\alpha} [zF(z)]'\alpha}{G^{1-\alpha}(z)\, G^{\alpha}(z)}$$

equivalently,

$$[zF'(z)]^{(1-\alpha)/\alpha} [zF'(z)]' = P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z). \quad \text{...(8)}$$

Also we have,

$$zF''(z) + F'(z) = \frac{P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z)}{[zF'(z)]^{(1/\alpha)-1}}. \quad \text{...(9)}$$

From the representation (5) for $F'(z)$ we have

$$z^{1/\alpha} F'(z)^{1/\alpha} = \frac{2}{\alpha} \int_0^z t^{(2-\alpha)/\alpha} f'^{1/\alpha}(t)\, dt$$

which on differentiation with respect to $z$ yields,

$$z^{1/\alpha} \cdot \frac{1}{\alpha} F'(z)^{(1-\alpha)/\alpha} F''(z) + F'^{1/\alpha}(z) \frac{2}{\alpha} z^{(2-\alpha)/\alpha}$$

$$= \frac{2}{\alpha} z^{(2-\alpha)/\alpha} f'^{1/\alpha}(z).$$

Hence,

$$F^{(1-\alpha)/\alpha}(z)\, [zF''(z) + 2F'(z)] = 2f'^{1/\alpha}(z).$$

Using (9) we get,

$$F'^{(1-\alpha)/\alpha}(z) \left\{ \frac{P^{1/\alpha}(z)\, G^{(1-\alpha)\alpha}(z)\, G'(z)}{[zF(z)]^{(1-\alpha)/1}} + F'(z) \right\} = 2f'^{(1-\alpha)/\alpha}(z)$$

or

$$\frac{P^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z)}{z^{(1-\alpha)/\alpha}} + F'^{1/\alpha}(z) = 2f'^{1/\alpha}(z).$$

Therefore,

$$zP'^{1/\alpha}(z)\, G^{(1-\alpha)/\alpha}(z)\, G'(z) + (zF'(z))^{1/\alpha} = 2\,[zf'(z)]^{1/\alpha}. \quad \text{...(10)}$$

Differentiating (10) with respect to $z$ and simplifying using (8) and (6) we get

$$\frac{1}{\alpha} P^{1/\alpha} g^{(1-\alpha)/\alpha} g' + \frac{z}{\alpha} P^{(1-\alpha)/\alpha} P' G^{(1-\alpha)/\alpha} G' = \frac{2}{\alpha} (zf')^{(1-\alpha)/\alpha} (zf')'.$$

Therfore

$$P^{1/\alpha} + \frac{zP^{(1-\alpha)/\alpha} P'}{\left(\frac{g^{(1-\alpha)/\alpha} g'}{G^{(1-\alpha)/\alpha} G'}\right)} = 2 \frac{(zf')^{(1-\alpha)/\alpha} (zf')'}{g^{(1-\alpha)/\alpha} g'}. \qquad \text{.. (11)}$$

Setting

$$P_1(z) = P^{1/\alpha}(z); \; Q(z) = \frac{g^{(1-\alpha)/\alpha}(z)\, g'(z)}{G^{(1-\alpha)/\alpha}(z)\, G'(z)}$$

(11) becomes

$$P_1(z) + \frac{\alpha z P_1'(z)}{Q(z)} = \frac{2\,(zf')^{(1-\alpha)/\alpha} (zf')'}{g^{(1-\alpha)/\alpha} g'}.$$

From (6) we get

$$Q(z) = \alpha \left( 1 + \frac{zG''(z)}{G'(z)} \right) + (1-\alpha) \frac{zG'(z)}{G(z)} + 1$$

since $G \in M_\alpha$ we find $\operatorname{Re} Q(z) > 0$. Also as $f \in Q(\alpha)$ we have

$$\left| \arg \left( P_1(z) + \frac{\alpha z P_1'(z)}{Q(z)} \right) \right| \leqslant \frac{\pi}{2\alpha}$$

an application of Theorem A for $\alpha \geqslant 1$ yields:

$$|\arg P_1(z)| \leqslant \frac{\pi}{2\alpha}; \text{ or } |\arg P(z)| \leqslant \frac{\pi}{2}$$

which gives $F \in Q(\alpha)$ thereby establishing the theorem.

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# EDGE CRACK IN ORTHOTROPIC ELASTIC HALF-PLANE

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Problems of determining the distribution of stress and displacement in an orthotropic elastic half-plane containing an edge crack normal to the free-surface when (I) the shape of the crack is prescribed and (II) the cracks are opened by given normal pressure, have been considered. Numerical results, for various loading functions, of stress intensity factor and crack energy have been presented for problem II taking into account the values of constants for boron-epoxy composite.

## 1. INTRODUCTION

The problems of determining stress and displacement fields in an isotropic elastic half-plane containing an edge crack have been considered by Sneddon and Das[10], Krapkhov[5], Srivastav and Narain[12], Irwin[3], Koiter[4] and many others. Details of references can be had in Sih[9], Sneddon and Lowengrub[11]. The authors used different techniques like integral transform method, complex variable method, alternating method, Wiener-Hopf technique etc. As such crack problems in orthotropic medium are important and useful from technological point of view. Dhaliwal[2], Satapathy and Parhi[8], Kushwaha[6] and Das and Behera[1] have considered Griffith cracks in orthotropic medium. In this paper we have dealt with two edge crack problems normal to the boundary in an orthotropic elastic half plane :

(I) having prescribed crack shape and the other, (II) having been opened by prescribed normal pressures.

The displacements and stresses in a two dimensional orthotropic elastic medium are expressed in terms of two potential functions[2,8], which are harmonic in two different planes both being different from the physical plane considered. Using integral transform technique closed form solution is obtained for problem I whereas the solution of problem II have been reduced in solving Fredholm integral equation of second kind which is suitable for numerical computation.

In problem I, expression for pressure necessary to produce the Griffith crack of prescribed shape have been obtained. Also in problem II, expressions for quantities of physical interest e.g. shape of the crack, stress intensity factor and crack energy have been obtained.

Numerical computations for problem II have been done for boron-epoxy composite material for linearly loaded, partially constant loaded, point loaded and constant loaded cracks.

## 2. The Basic Equations

For an orthotropic medium, we choose the Cartesian co-ordinate axes to be coincident with the principal material axes. For the case of plane strain following Satapathy and Parhi[8] the displacement and stress components are expressed in terms of two potential functions $\varphi_1$ and $\varphi_2$ as

$$u_x = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial x} \qquad \ldots(2.1)$$

$$u_y = \frac{\lambda_1}{\delta_1} \frac{\partial \varphi_1}{\partial y_1} + \frac{\lambda_2}{\delta_2} \frac{\partial \varphi_2}{\partial y_2} \qquad \ldots(2.2)$$

$$\sigma_{xx}/A_{66} = \frac{1+\lambda_1}{\delta_1^2} \frac{\partial^2 \varphi_1}{\partial x^2} + \frac{1+\lambda_2}{\delta_2^2} \frac{\partial^2 \varphi_2}{\partial x^2} \qquad \ldots(2.3)$$

$$\sigma_{yy}/A_{66} = (1+\lambda_1) \frac{\partial^2 \varphi_1}{\partial y_1^2} + (1+\lambda_2) \frac{\partial^2 \varphi_2}{\partial y_2^2} \qquad \ldots(2.4)$$

$$\sigma_{xy}/A_{66} = \frac{1+\lambda_1}{\delta_1} \frac{\partial^2 \varphi_1}{\partial x\, \partial y_1} + \frac{1+\lambda_2}{\delta_2} \frac{\partial^2 \varphi_2}{\partial x\, \partial y_2} \qquad \ldots(2.5)$$

$\varphi_1$ and $\varphi_2$ satisfy the differential equation

$$\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial y_i^2} = 0, \; (i = 1, 2) \qquad \ldots(2.6)$$

where

$$y_i = y/\delta_i \; (i = 1, 2) \qquad \ldots(2.7)$$

and $\lambda_1$, $\lambda_2$ and $\delta_1^2$, $\delta_2^2$ are the roots of the quadratic equations

$$\lambda^2 A_{66}\,(A_{12} + A_{66}) + \lambda\,[(A_{12} + A_{66})^2 + A_{66}^2 - A_{11} A_{22}] + A_{66}\,(A_{12} + A_{66}) = 0 \qquad \ldots(2.8)$$

and

$$\delta^4 A_{11} A_{66} + \delta^2\,[(A_{12} + A_{66})^2 - A_{66}^2 - A_{11} A_{12}] + A_{22} A_{66} = 0 \qquad \ldots(2.9)$$

respectively. $A_{ij}$ are anisotropic constants of the orthotropic material.

## 3. Formulation of the Problems

We consider the semi-infinite region $x \geqslant 0$ and take the half plane to be orthotropic containing a Griffith edge crack $0 \leqslant x \leqslant a$ on $y = 0$ normal to the free surface

$x = 0$. Due to symmetry with respect to the $x$-axis the problems are reduced to quarter plane ($x > 0$, $y > 0$) problems.

*Problem I (Displacement prescribed)*

The problem is to determine the components of displacements and stresses at any point and to determine the pressure $p(x)$ necessary to produce the crack of prescribed shape $w(x)$. The boundary conditions of the problem are

$$u_y(x, 0) = w(x), \; 0 \leqslant x \leqslant a \qquad \ldots(3.1)$$

$$u_y(x, 0) = 0, \; x > a \qquad \ldots(3.2)$$

$$\sigma_{xy}(x, 0) = 0, \; x > 0 \qquad \ldots(3.3)$$

$$\sigma_{xx}(0, y) = 0, \; y \geqslant 0 \qquad \ldots(3.4)$$

and

$$\sigma_{xy}(0, y) = 0, \; y \geqslant 0. \qquad \ldots(3.5)$$

*Problem II (Stress prescribed)*

The crack is opened by equal and opposite prescribed pressures $p_0 f(x)$ normal to the faces of it. The problem is to determine the components of displacements and stresses at any point and to determine the shape of the crack, stress intensity factor at the crack tip and the crack energy. The boundary conditions of the problem are

$$\sigma_{yy}(x, 0) = -p_0 f(x), \; 0 \leqslant x \leqslant a \qquad \ldots(3.6)$$

together with (3.2) to (3.5).

## 4. Solution of the Problems

We take solutions of eqns. (2.6) in the form

$$\varphi_1(x, y) = \int_0^\infty [\alpha^{-1} B_1(\alpha) e^{-\alpha \delta_1 x} \cos \alpha y$$

$$+ \alpha^{-2} C_1(\alpha) e^{-\alpha y_1} \cos \alpha x] \, d\alpha \qquad \ldots(4.1)$$

and

$$\varphi_2(x, y) = \int_0^\infty [\alpha^{-1} B_2(\alpha) e^{-\alpha \delta_2 x} \cos \alpha y$$

$$+ \alpha^{-2} C_2(\alpha) e^{-\alpha y_2} \cos \alpha x] \, d\alpha \qquad \ldots(4.2)$$

where $B_1(\alpha)$, $B_2(\alpha)$, $C_1(\alpha)$ and $C_2(\alpha)$ are unknown functions to be determined. Boundary conditions (3.3) and (3.5) will be satisfied, if we take

$$\frac{1 + \lambda_1}{\delta_1} C_1(\alpha) = - \frac{1 + \lambda^2}{\delta_2} C_2(\alpha) \qquad \ldots(4.3)$$

and

$$B_1(\alpha) = -\frac{1+\lambda_2}{1+\lambda_1}\,\frac{\delta_2}{\delta_1}\,B_2(\alpha). \qquad \text{...(4.4)}$$

From boundary conditions (3.2) and (3.4), using (4.3) and (4.4), we obtain

$$\int_0^\infty \alpha^{-1} C_2(\alpha)\cos\alpha x\,d\alpha = 0, \quad \text{for all } x > a \qquad \text{...(4.5)}$$

$$\frac{(\delta_1-\delta_2)}{\delta_1}\int_0^\infty B_2(\alpha)\,\alpha\cos\alpha y\,d\alpha$$

$$+\frac{1}{\delta_2}\int_0^\infty\left[\frac{e^{-\alpha y_1}}{\delta_1}-\frac{e^{-\alpha y_2}}{\delta_2}\right]C_2(\alpha)\,d\alpha = 0,\ y \geqslant 0. \qquad \text{...(4.6)}$$

*Problem I*

Boundary condition (3.1) with the help of (4.3) gives

$$\int_0^\infty C_2(\alpha)\,\alpha^{-1}\cos\alpha x\,d\alpha = \theta(x),\ 0 \leqslant x \leqslant a \qquad \text{...(4.7)}$$

where

$$\theta(x) = \frac{(1+\lambda_1)\,\delta_2}{(\lambda_1-\lambda_2)}\,w(x). \qquad \text{...(4.8)}$$

Now eqns. (4.5) – (4.7) will determine the unknown functions $B_2(\alpha)$ and $C_2(\alpha)$. Taking

$$C_2(\alpha) = \alpha\int_0^a t\,\varphi(t)\,J_0(\alpha t)\,dt$$

$$B_2(\alpha) = \int_0^\infty t\,\psi(t)\,J_0(\alpha t)\,dt \qquad \text{...(4.9)}$$

eqn. (4.5) is identically satisfied and eqn. (4.7)

leads to (c. f. Magnus and Oberhettinger[7])

$$\int_x^a \frac{t\,\varphi(t)\,dt}{(t^2-x^2)^{1/2}} = \theta(x),\ 0 \leqslant x \leqslant a$$

which has the solution

$$\varphi(t) = \frac{2}{\pi}\left[\frac{\theta(a)}{(a^2-t^2)^{1/2}} - \int_t^a \frac{\theta'(u)\,du}{(u^2-t^2)^{1/2}}\right],\ 0 < t \leqslant a. \qquad \text{...(4.10)}$$

Equation (4.6) with the help of (4.9) and (4.10) gives

$$\int_0^y \frac{t\,\psi(t)\,dt}{(y^2-t^2)^{1/2}} = \frac{2\delta_1}{\pi\delta_2(\delta_1-\delta_2)} \int_0^a u\left[\frac{1}{\left(y_1^2+u^2\right)^{1/2}}\right.$$

$$\left.-\frac{1}{\left(y_2^2+u^2\right)^{1/2}}\right]\times\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}} - \int_u^a \frac{\theta'(\beta)\,d\beta}{(\beta^2-u^2)^{1/2}}\right]du$$

which has the solution

$$t\,\psi(t) = \left(\frac{2}{\pi}\right)^2 \frac{\delta_1}{\delta_2(\delta_1-\delta_2)} \int_0^a u\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}} - \int_u^a \frac{\theta'(\beta)d\beta}{(\beta^2-u^2)^{1/2}}\right]$$

$$\times\left[\frac{d}{dt}\int_0^t \frac{y}{(t^2-y^2)^{1/2}}\left(\frac{1}{\left(y_1^2+u^2\right)^{1/2}}\right.\right.$$

$$\left.\left.-\frac{1}{(y_2^2+u^2)^{1/2}}\right)dy\right]du.$$

Simplifying we get,

$$\psi(t) = \frac{4\delta_1 t}{\pi^2\,\delta_2\,(\delta_1-\delta_2)} \int_0^a \left[\frac{1}{t^2+u^2\,\delta_2^2} - \frac{1}{t^2+u^2\,\delta_1^2}\right]$$

$$\times\left[\frac{\theta(a)}{(a^2-u^2)^{1/2}} - \int_u^a \frac{\theta'(\beta)\,d\beta}{(\beta^2-u^2)^{1/2}}\right]du,\ t > 0. \qquad \ldots(4.11)$$

Equation (4.10) and (4.11) give $\psi(t)$ and $\phi(t)$ in closed form. So $B_2(\alpha)$ and $C_2(\alpha)$ are known from (4.9). Then (4.3) and (4.4) give $C_1(\alpha)$ and $B_1(\alpha)$ respectively. Therefore the displacements and stresses can be determined from (2.1) – (2.5).

Expression for pressure $p(x)$ necessary to produce the Griffith crack of prescribed shape is obtained from (2.4) with the help of (4.1) – (4.4) and (4.9) as

$$p(x) = A_{66}(1+\lambda_2)\left[\delta_2\,x\int_0^\infty\left\{\frac{\delta_1^2}{(t^2+\delta_1^2\,x^2)^{3/2}} - \frac{\delta_2^2}{(t^2+\delta_2^2\,x^2)^{3/2}}\right\}\right.$$

$$\left.\times\, t\,\psi(t)\,dt + \left(1-\frac{\delta_1}{\delta_2}\right)\frac{d}{dx}\int_0^x \frac{t\,\varphi(t)\,dt}{(x^2-t^2)^{1/2}}\right],\ 0 \leqslant x \leqslant a.$$

*Problem II*

Boundary condition (3.6) with the help of (4.3) and (4.4) gives

$$\delta_2 \int_0^\infty \alpha\, B_2(\alpha)\, [\delta_1\, e^{-\alpha \delta_1 x} - \delta_2\, e^{-\alpha \delta_2 x}]\, d\alpha + \left(1 - \frac{\delta_1}{\delta_2}\right)$$

$$\times \int_0^\infty C_2(\alpha) \cos \alpha x\, d\alpha = \frac{-p_0 f(x)}{A_{66}(1 + \lambda_2)},\ 0 \leqslant x \leqslant a. \qquad \text{...(4.12)}$$

In this case, (4.5), (4.6) and (4.12) will determine the unknown functions $B_2(\alpha)$ and $C_2(\alpha)$. Taking the same integral representations given in (4.9) for $B_2\ \alpha$ and $C_2(\alpha)$ as in problem I, eqn. (4.5) is identically satisfied and eqn. (4.6) leads to

$$\int_0^y \frac{t\, \psi(t)\, dt}{(y^2 - t^2)^{1/2}} = \frac{\delta_1}{\delta_2(\delta_1 - \delta_2)} \int_0^a u\, \varphi(u)$$

$$\times \left[\frac{1}{(y_1^2 + u^1)^{1/2}} - \frac{1}{(y_2^2 + u^2)^{1/2}}\right] du. \qquad \text{...(4.13)}$$

Inverting (4.13) we get

$$\psi(t) = \int_0^a K_2(t, u)\ \varphi(u)\, du,\ t \geqslant 0 \qquad \text{...(4.14)}$$

where

$$K_2(t, u) = \frac{2\delta_1 t}{\pi\, \delta_2(\delta_1 - \delta_2)} \left[\frac{1}{t^2 + u^2 \delta_2^2} - \frac{1}{t^2 + u^2 \delta_1^2}\right]. \qquad \text{...(4.15)}$$

Equation (4.12) under (4.9) becomes

$$\int_0^x \frac{t\, \varphi(t)\, dt}{(x^2 - t^2)^{1/2}} = \frac{\delta_2}{(\delta_1 - \delta_2)} \left[\delta_2 \int_0^\infty u\, \psi(u) \left\{\frac{1}{(\delta_2^2 x^2 + u^2)^{1/2}}\right.\right.$$

$$\left.\left. - \frac{1}{(\delta_1^2 x^2 + u^2)^{1/2}}\right\} du + \frac{p_0\, F(x)}{A_{66}(1 + \lambda_2)}\right] \qquad \text{...(4.16)}$$

where

$$F(x) = \int_0^x f(x)\, dx. \qquad \text{...(4.17)}$$

Inverting (4.16) we get

$$\varphi(t) = \int_0^\infty K_3(t, u)\, \psi(u)\, du + F_1(t),\ 0 \leqslant t \leqslant a \qquad \text{...(4.18)}$$

where

$$K_3(t, u) = \frac{2}{\pi} \frac{\delta_2^2 t}{(\delta_1 - \delta_2)} \left[ \frac{\delta_1^2}{t^2 \delta_1^2 + u^2} - \frac{\delta_2^2}{t^2 \delta_2^2 + u^2} \right] \quad \ldots(4.19)$$

and

$$F_1(t) = \frac{2}{\pi} \frac{\delta_2 p_0}{(\delta_1 - \delta_2) A_{66} (1 + \lambda_2)} \int_0^t \frac{f(x)\, dx}{(t^2 - x^2)^{1/2}}. \quad \ldots(4.20)$$

The solution of the problem, therefore, reduces to the solution of the pair of simultaneous integral eqns. (4.14) and (4.18).

Eliminating $\psi(t)$ from (4.14) and (4.18) we get

$$\varphi(t) - \int_0^a \varphi(v) L(t, v)\, dv = F_1(t), \; 0 \leqslant t \leqslant a \quad \ldots(4.21)$$

where

$$L(t, v) = \frac{4 \delta_1 \delta_2 t}{\pi^2 (\delta_1 - \delta_2)^2} \left[ \frac{\delta_1^2}{(t^2 \delta_1^2 - v^2 \delta_2^2)} \log\left(\frac{t \delta_1}{v \delta_2}\right) + \frac{\delta_2^2}{(t^2 \delta_2^2 - v^2 \delta_1^2)} \log\left(\frac{t \delta_2}{v \delta_1}\right) - \frac{2}{(t^2 - v^2)} \times \log\left(\frac{t}{v}\right) \right]. \quad \ldots(4.22)$$

This kernel $L(t, v)$ in eqn. (4.22) appears to have singularities at points $t = v\, \delta_2/\delta_1$, $v\, \delta_1/\delta_2$ and $t = v$. But it can be easily shown by $L'$ Hospital's rule that $L(t, v)$ tends to finite limits when $t$ tends to these three points. Accordingly, eqn. (4.21) is suitable for its numerical solution. Thus quantities of physical interest of the problem can be determined. The stress intensity factor $K_I$ is given by

$$K_I = \lim_{x \to a+} [2(x - a)]^{1/2} \sigma_{yy}(x, 0)$$

$$= \frac{A_{66}(1 + \lambda_2)(\delta_1 - \delta_2)}{\delta_2} a^{1/2} \varphi(a). \quad \ldots(4.23)$$

Crack energy $W$ is given by

$$W = - \int_0^a \sigma_{yy}(x, 0)\, u_y(x, 0)\, dx$$

$$= \frac{p_0 (\lambda_1 - \lambda_2)}{(1 + \lambda_1)\, \delta_2} \int_0^a t\, \varphi(t) \left\{ \int_0^t \frac{f(x)\, dx}{(t^2 - x^2)^{1/2}} \right\} dt. \quad \ldots(4.24)$$

The shape of the crack is obtained from (2.2) using (2.8), (4.3) and (4.9) as

$$u_y(x, 0) = \frac{(1-\lambda_2)}{\delta^2} \int_x^a \frac{t\ \varphi(t)\, dt}{(t^2 - x^2)^{1/2}}, \quad 0 \leqslant x \leqslant a. \qquad \ldots(4.25)$$

Taking the values of the elastic constants as

$$A_{66} = 1.13 \times 10^6 \text{ psi}, \lambda_1 = 17.430, \lambda_2 = 0.057.$$

$\delta_1^2 = 2.862$ and $\delta_2^2 = 0\ 047$ for Boron-Epoxy composite (c.f. Das and Behera[1], $K_I$, $W$ and $\varphi(t)$ in (4.21) for different $t$, for unit crack length have been computed for various loading functions $f(x)$. The results obtained are given in Table I and presented graphically in Fig. 1 respectively.

TABLE I

| *Loading function* $f(x)$ | *Stress intensity factor* ($K_I$) | *Crack energy* ($W$) |
|---|---|---|
| 1 | 1.06893 $p_0$ | 0.23598 $p_0^2$ |
| $H(x - \frac{1}{2})$ | 1.04894 $p_0$ | 0.33632 $p_0^2$ |
| $x$ | 0.66221 $p_0$ | 0.09149 $p_0^2$ |
| $\delta(x - 1/2)$ | 0.68220 $p_0$ | 0.60762 $p_0^2$ |

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# FLOW OF A SECOND ORDER FLUID DUE TO THE ROTATION OF AN INFINITE POROUS DISK NEAR A STATIONARY PARALLEL POROUS DISK

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Here the flow pattern of an incompressible second order fluid confined between two infinite parallel disks, one stationary (stator) and other rotating (rotor), has been considered when there is a uniform injection normal to the stator. The solution of the problem is sought by expanding all the flow functions in the ascending powers of Reynolds numbers. The effects of the elastico-viscosity, cross-viscosity and injection are studied on the flow when any two of them are kept fixed and the third is varied.

## 1. INTRODUCTION

Recently, there has been a considerable amount of interest in the flow of non-Newtonian fluids between rotating disks, since the flow geometry is one which has several technical applications, for example lubrication. Consequently a number of workers have studied the flows between two disks under various physical and experimental conditions.

Recently, Batchelor[1] gave a note on a class of solutions of the Navier and Stokes equations representing steady rotationally symmetric flow. Mellor *et al.*[2] studied the flow of an incompressible, viscous fluid between stationary and rotating disks and presented numerical evidence that for a single Reynolds number many steady state solutions are possible. Soundalgekar[3] considered the flow of a second order viscous fluid in a circular tube under the pressure gradient varying exponentially with time. Similarly, Rajgopal[4] studied the flow of a second order fluid between rotating parallel plates. Lai *et al.*[5] studied the flows occurring between the parallel rotating disks to include the solutions that are not axi-symmetric. Lai *et al.*[6] geteralized the Von-Karman solution for flow above a single rotating disk. Szeri and Rajgopal[7] discussed the flow of a fluid of grade three between heated parallel plates. Christie *et al.*[8] made the study of the flow of a non-Newtonian fluid between concentric rotating cylinders, More recently Sirivat *et al.*[6] made an experimental investigation of the results obtained from the flow of non-Newtonian fluid between rotating parallel disks.

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The authors were motivated by the problems tackled by Wilson and Schryer[10] and Sharma and Singh[11]. Wilson and Schryer considered the viscous flow between two co-axial infinite disks, one stationary and other rotating. They took also into consideration the effects of applying a uniform suction through the rotating disk. Sharma and Singh, on the other hand, considered the flow of a second order incompressible fluid due to torsional oscillations of infinite disks. They also took the effects of applying uniform injection to lower disk and an equal suction to the upper one. Moreover, both the problems were for unsteady flows. In our present problem, we take the steady flow into account deleting the torsional effects of oscillations. We here study the flow pattern of an incompressible second order fluid between two parallel infinite disks when one is rotating (called rotor) and other is at rest (called stator). A uniform injection is applied to the stator forming the subject matter of the paper. The remaining conditions are the same, i. e., the rotor coincides with the plane $z = 0$ and the stator coincides with the plane $z = d$. Here the dimensionless parameters $\alpha$ $\left(= \frac{\mu_2}{\rho d^2}\right)$, $\beta \left(= \frac{\mu_3}{\rho d^2}\right)$ govern the effects of elastico-viscosity and cross-viscosity, while the effects of injection are governed by a non-dimensional parameter $k \left(= \frac{\omega_0}{2d\Omega}\right)$ where $\omega_0$ is the suction velocity (negative for injection).

## 2. Basic Equations

The constitutive equations of an incompressible second order fluid are (cf. Srivastava[12])

$$\tau_{ij} = p\,\delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)ik} A_{(1)kj} \quad \text{...(1)}$$

$$A_{(1)ij} = v_{i,j} + v_{j,i}, \; A_{(2)ij} = a_{i,j} + a_{j,i} + 2 v_{m,i}\, v_{m,j}. \quad \text{...(2)}$$

Here $\tau_{ij}$ is the stress tensor, $p$ hydrostatic pressure, $v_i$ and $a_i$ are the velocity and the acceleratinn vectors; $\mu_1$, $\mu_2$, $\mu_3$ are the mathematical constants and $\delta_{ij}$ is Kronecker's delta.

The second order fluid characterised by equation (1) confined between two infinite disks. The disk coinciding with the plane $z = 0$ rotates with a uniform angular velocity $\Omega$ about $z$-axis, while the other (assumed porous) coincides with the plane $z = d$ and is at rest. A uniform injection $-w_0$ ($w_0$ being positive in $z$-increasing direction) is applied normal to the rotating disk. The space between the disks is occupied by homogeneous, incompressible second order fluid. If we choose the cylindrical co-ordinate system as $(r, \theta, z)$, then the equation (1) together with momentum equation for extraneous force are (cf. Srivastava[12] and Wilson and Schryer[10])

$$\rho\left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} \quad \text{...(3)}$$

$$\rho\left(u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{u v}{r}\right) = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\,\tau_{\theta r}}{r} \quad \text{...(4)}$$

$$\rho \left( u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) = \frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} \qquad ...(5)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad ...(6)$$

where $u$, $v$ and $w$ are the radial, transverse and axial components of velocity respectively. the relevant boundary conditions of the problem are

$$u = 0,\ v = r\,\Omega,\ w = 0 \text{ at } z = 0$$

$$n = 0,\ v = 0,\ w = -\ w_0 \text{ at } z = d. \qquad ...(7)$$

The velocity components for axi-symmetric flow compatible with continuity criterion (6) can be taken as (cf. Von-Karman[13])

$$u = r\ \Omega\ F'(\eta)$$

$$v = r\,\Omega\ F'(\eta)$$

$$w = -\ 2d\ \Omega\ F(\eta) \qquad ...(8)$$

and

$$P = \mu_1\ \Omega \left[ -\ p_1(\eta) + \frac{r^2}{d^2} \{(B - 2A)(F''^2 + G'^2) + \lambda\} \right] \qquad ...(9)$$

where

$$\eta = z/\alpha,\ A = \frac{\mu_2\ \Omega}{\mu_1},\ B = \frac{\mu_3\ \Omega}{\mu_1}.$$

Here the primes denote derivatives w.r.t. $\eta$ and $\lambda$ is an arbitrary constant to be determined from the boundary conditions (7).

In view of (8), the boundary conditions (7) become

$$\eta = 0:\ F = 0 = F',\ G = 1$$

$$\eta = 1:\ F = k,\ F' = 0,\ G = 0. \qquad ...(10)$$

Following set of the equations is obtained after substituting (8) and (9) into (1), (3), (4), (5) (cf. Srivastava[12])

$$R\,(F'^2 - G^2 - 2FF'') = F''' + 2\alpha\, R\,(F''^2 + 2G'^2 + FF^{iv})$$

$$-\beta\ R\ (F''^2 + 3G'^2 + 2F'\ F''') - 2\lambda \qquad ...(11)$$

$$2R\,(F'\,G - FG') = G'' - 2\alpha\ R\,(F''\,G' - F\,G'') + 2\,\beta\ R\,(F''\,G' - F''\,G'') \qquad ...(12)$$

$$4R\ FF' = p_1' - 2F'' - 4\alpha\ R\,(11F'\ F'' + FF''') + 28\ \beta\ R\ F'\ F'' \qquad ...(13)$$

where

$$\alpha = \frac{\mu_2}{\rho d^2}, \beta = \frac{\mu_3}{\rho\, d^2}, R = \frac{d^2\,\Omega\,\rho}{\mu_1}$$

and primes denote differentiation w.r.t. $\eta$. From (13), we can get $p_1$ after integration and substituting it in (9), we get hydrosatic pressure in terms of $\bar{F}$, $G$, $\lambda$ and their derivatives Here one thing is to be noted that the equations (11), (12) and (13) can also be obtained from Sharma and Singh[14] and Wilson and Schryer[10] who studied the same problem under different boundary conditions after deleting the terms containing time.

## 3. Solutions and Equations

The solutions of (11) — (13) cannot be found for all values of $R$ with the help of a regular perturbation technique. For finite values of $R$ it is not possible to derive even an approximation solution. So, one of the possible solutions is to seek it for values of $R$.

Taking $R$ to be small, we expand $F$, $G$ and $\lambda$ in ascending powers of $R$ as follows:

$$F = f_0 + Rf_1 + R^2 f_2 + \ldots$$

$$G = g_0 + Rg_1 + R^2 g_2 + \ldots$$

$$\lambda = \lambda_0 + R\lambda_1 + R^2 \lambda_2 + \ldots \qquad \ldots(14)$$

Substituting (14) in (11) and (12), we get sets of linear differential equations by equating similar powers of $R$ on both the sides and neglecting the powers higher than two,

$$f_0''' = 2\,\lambda_0,\ g_0'' = 0 \qquad \ldots(15)$$

$$f_1''' + 2\alpha\left(f_0'' + 2g_0' + f_0 f_0^{v}\right) - \beta\left(f_0''^2 + 3g_0'^2 + 2f_0'''\right)$$
$$= f_0'^2 - g_0^2 - 2f_0 f_0'' + 2\lambda_1 \qquad \ldots(16)$$

$$g_1''' - 2\alpha\left(f_0'' g_0' - f_0 g_0'''\right) + 2\beta\left(f_0'' g_0' - f_0' g_0''\right)$$
$$= 2\, f_0' g_0 - 2f_0\, g_0' \qquad \ldots(17)$$

$$f_2''' + 2\alpha\left(f_1'' + 2g_0' f_1' + f_0 f_1^{v} + f_1 f_0^{v}\right)$$
$$- \beta\left(2f_0'' f_1'' + 3g_0' g_1' f_0'''\right)$$
$$= 2f_0' f_1' - 2g_0 g_1 - 2f_0 f_1'' - 2f_1 f_0'' + 2\lambda_2 \qquad \ldots(18)$$

$$g_2'' - 2\alpha\left(f_0'' g_1' + f_1'' g_0' - f_0 g_0''' \; f_1 g_0'''\right) + 2\beta\left(f_0'' g_1'\right.$$

$$\left. + f_1'' g_0' - f_1' \; g_0'' - f_0' g_1' \right)$$

$$= f_0' \; g_1 + f_1' \; g_0 - f_0 g_1' + f_1 g_0' \; . \qquad \ldots(19)$$

The boundary conditions (10) become

$$f_n(0) = 0 = f_n'(0)$$

$$f_0(1) = k, \; g_0(0) = 1$$

$$f_{n+1}(1) = 0 = g_{n+1}(0)$$

$$g_n(1) = 0, \; (\text{for } n = 0, 1, 2, 3, \ldots) \qquad \ldots(20)$$

where $K = \dfrac{w_0}{2d\Omega}$ is a dimensionless parameter representing the injection of the stator.

On solving the sets of equations (15) — (19) under the boundary conditions (20), we can easily get $f_0, f_1, f_2, \ldots; g_0, g_1, g_2, \ldots$ and $\lambda_0, \lambda_1, \lambda_2, \ldots$. Substituting their

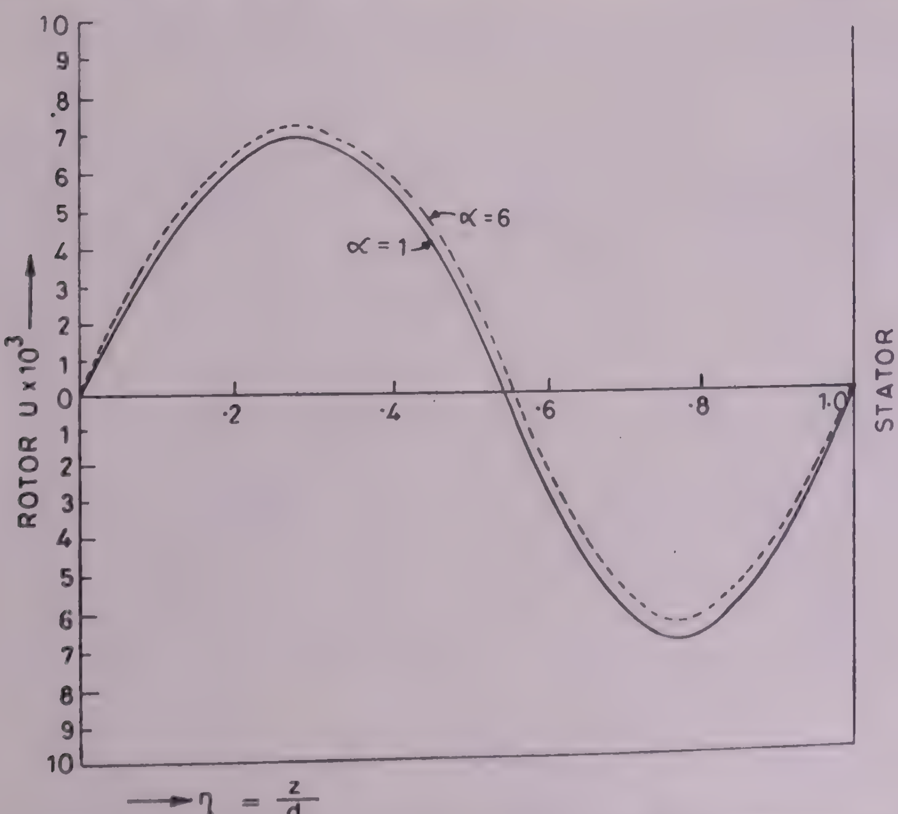


FIG. 1. Response of radial velocity to an increase in elastico-viscous effects $\beta = 10$, $K = .0002$.

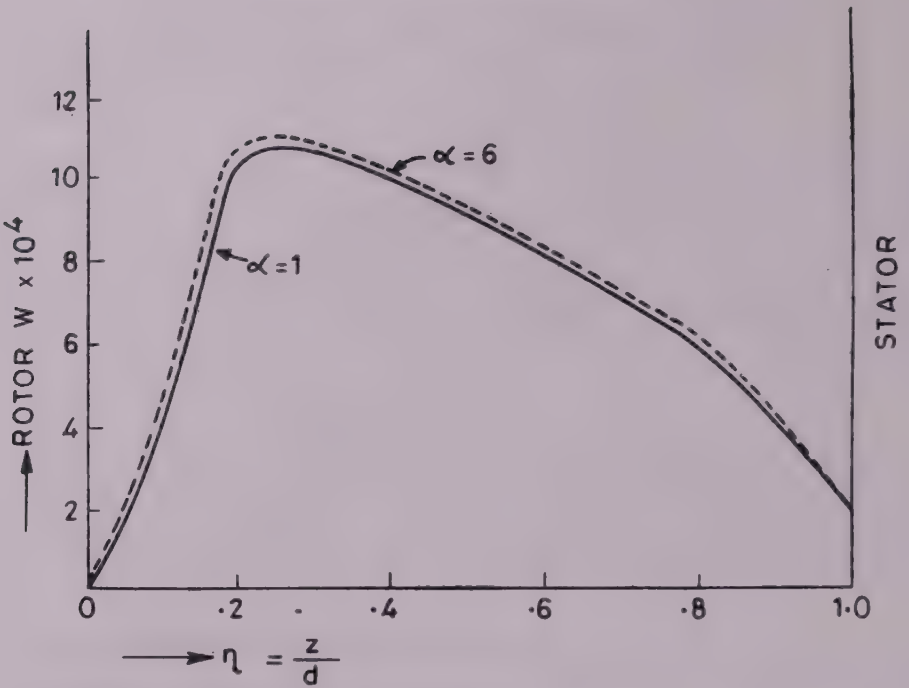


FIG. 2. Response of axial-velocity to an increase in elastico-viscous effects $\beta = 10$. $K = .0002$.

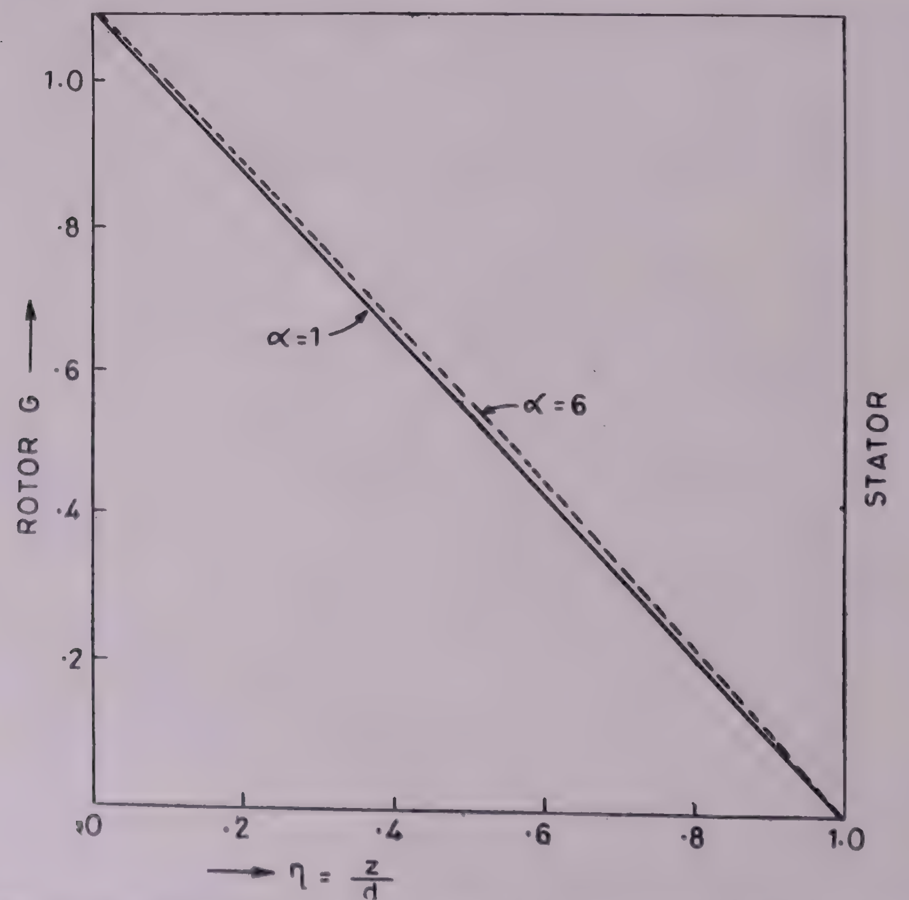


FIG. 3. Response of transverse velocity to an increase in elastico-viscous effects $\beta = 10$, $K = .0002$.

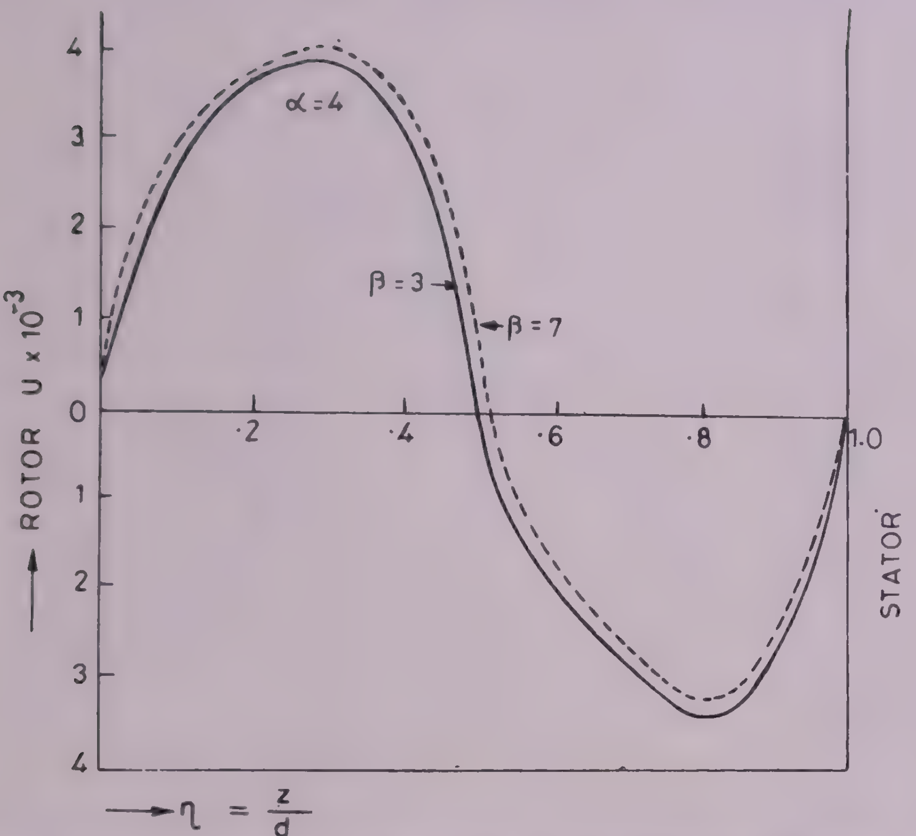


Fig. 4. Response of radial velocity to an increase in cross-viscous effects $\alpha=4$ and $K=.0002$.

values in (14) we find the expressions for $u$, $v$ and $w$. Since the method of solving the equations (15) —(19) is a straight forward procedure, hence there is not any problem in finding out the expressions for $f_0, f_1, f_2, \ldots$ ; $g_0, g_1, g_2, \ldots$ and $\lambda_0, \lambda_1, \lambda_2, \ldots$ Then with the help of (14) and (8) we get the expressions for $u$, $v$ and $w$.

## 4. Graphical Representation and Conclusions

Numerical computations are carried out for different values of $\alpha$, $\beta$, $k$ for $u$, $v$, $w$ and they are shown on figures. For each case, $R$ is assumed constant and equal to 0.5. The effect of an increase of elastico-viscous forces in the fluid is observed by maintaining cross-viscous ($\beta$) and injection parameter ($k$) as equal to 10 and 0.0002 respectively. It is found that, with an increase in elastico-viscosity of the fluid, the radial component of velocity increases and the transverse component of velocity decreases throughout the gap length. The axial velocity increases near the rotor and decreases near the stator as is evident from Figs 1–3.

In turn, if we choose to increase the cross-viscous forces keeping $\alpha$ and $k$ as equal to 4 and 0.0002 respectively, we find that the transverse component increases and the axial component decreases throughout the space between the disks. The radial

component decreases near the rotor and increases near the stator as is also evident Figs. 4–6.

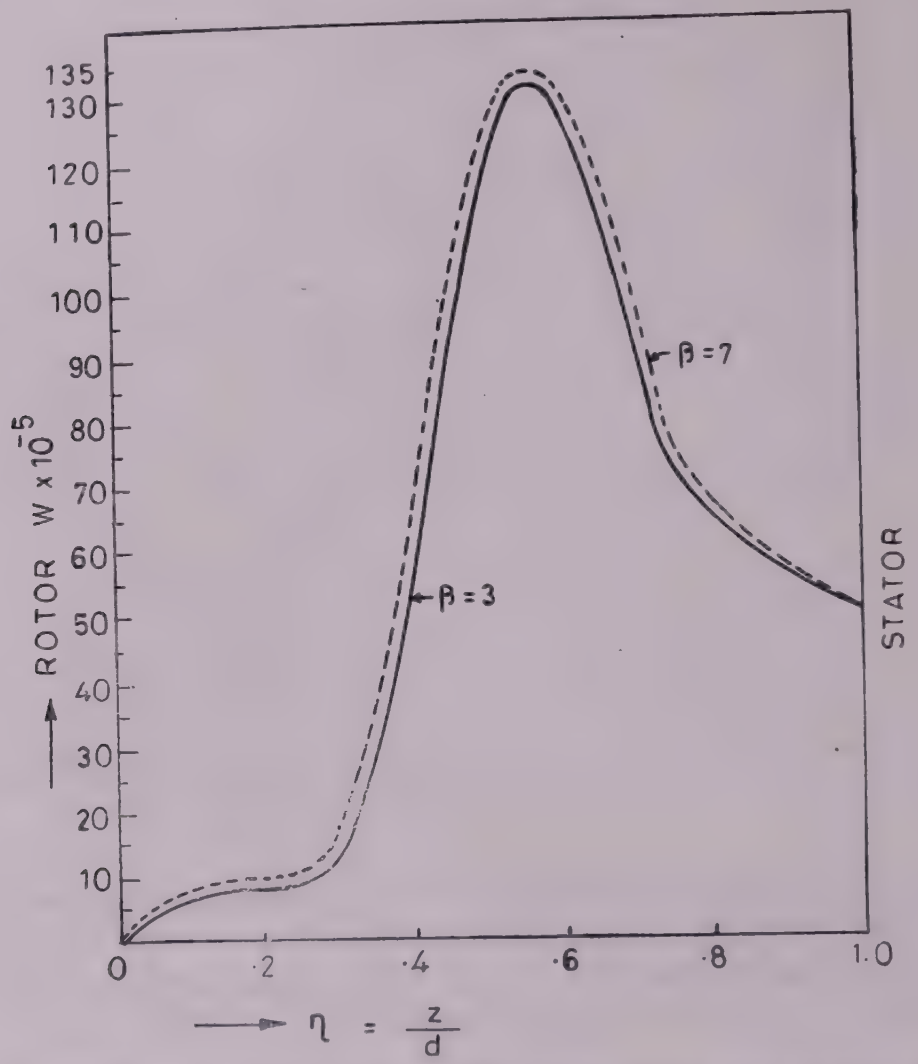


FIG. 5. Response of axial velocity to an increase in cross-viscous effects $\alpha=4$ and $K=.0002$.

The effect of an increase in injection is found by assuming $\alpha = 0.0276$ and $\beta = 0.1233$ so that $\alpha$ and $\beta$ are in the same ratio as $\mu_1$ and $\mu_2$ for 6.8 precent solutions of poly-iso-butylene incetane at 30°C. It is concluded that in such type of second order fluids, an increase in injection on the stator increases radial axial flow throughout the gap, while the transverse component behaves contrary to it as is evident from Figs. 7–9.

The results obtained to incompressible second order fluids can easily be compared with these of Newtonian fiuids. The non-Newtonian effects are exhibited through two non-dimensional parameters $\alpha$, $\beta$ called elastico-viscosity, cross-viscosity respectively. Thus for Newtonian fluids $\alpha = \beta = 0$. Taking $\alpha = \beta = 0$ in the expressions of $u$, $v$

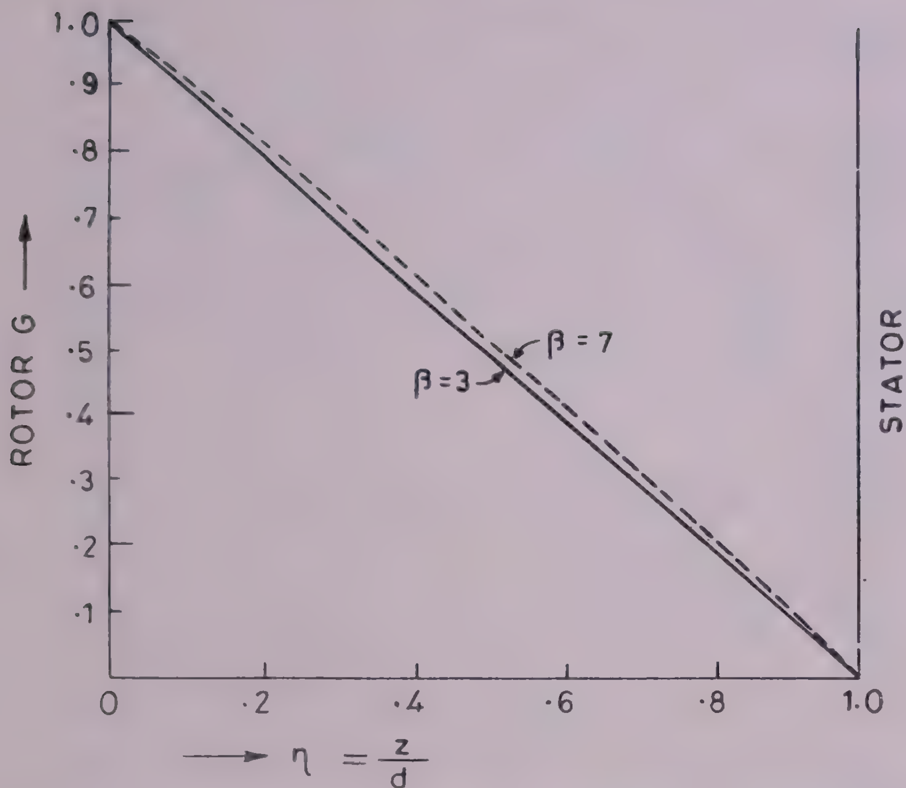


FIG. 6. Response of transvers velocity to an increase in cross-viscous effects $\alpha=4$ and $K=.0002$.

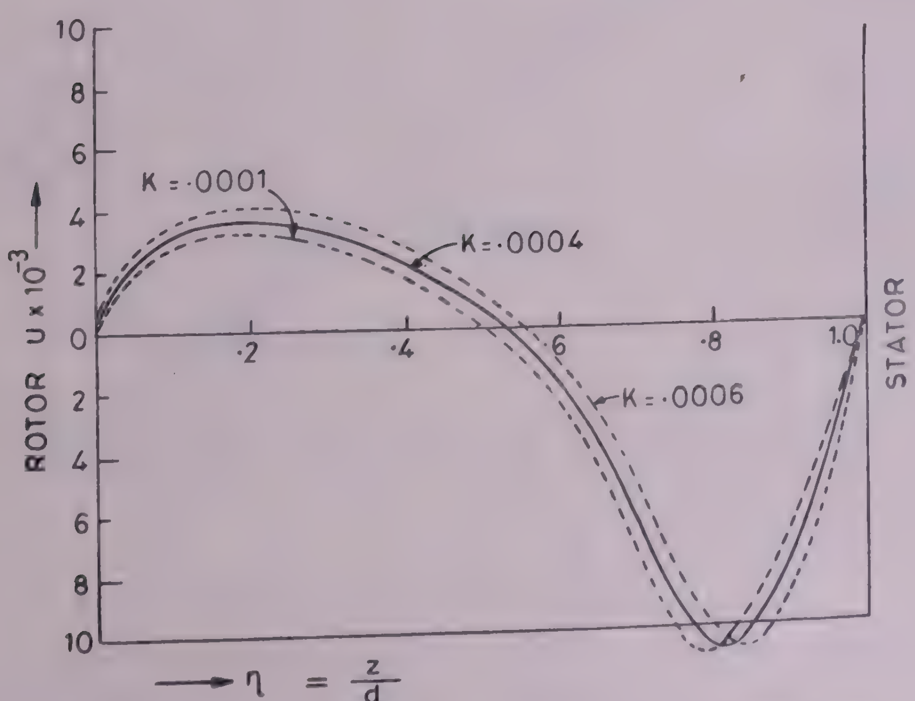


FIG. 7. Variation of radial velocity with injection $\alpha=.0276$ and $\beta=.1233$.

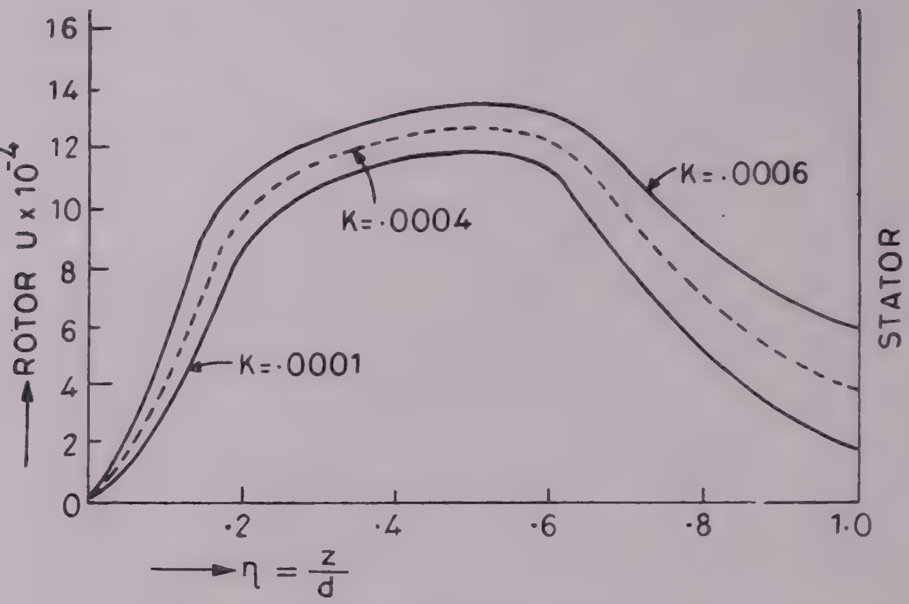


FIG. 8. Variation of axial velocity with injection $\alpha = .0276$ and $\beta = .1233$.

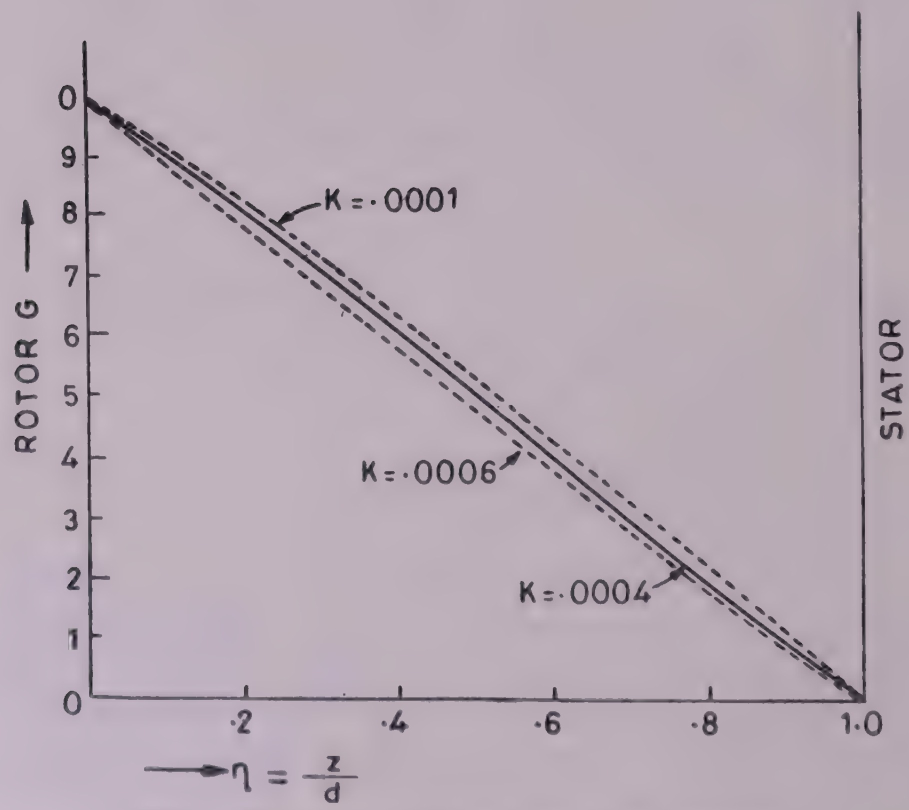


FIG. 9. Variation of transverse velocity with injection.

and $w$, we shall be able to find the corresponding expressions for Newtonian fluids. Here taking $\alpha$ and $\beta$ as fixed (i. e., zero) and varying $k$, we can easily study the effects on $u$, $v$ and $w$. The effects are scown in the Figs. 10-12.

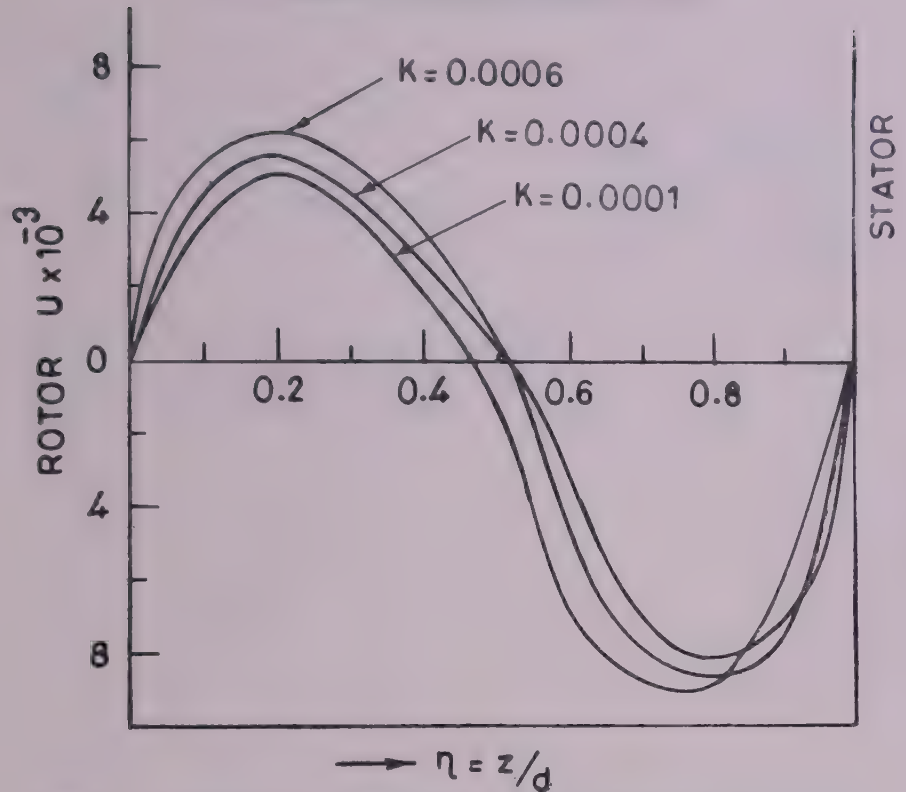


FIG. 10. Radial velocity with injection for newtonian fluid ($\alpha=\beta=0$).

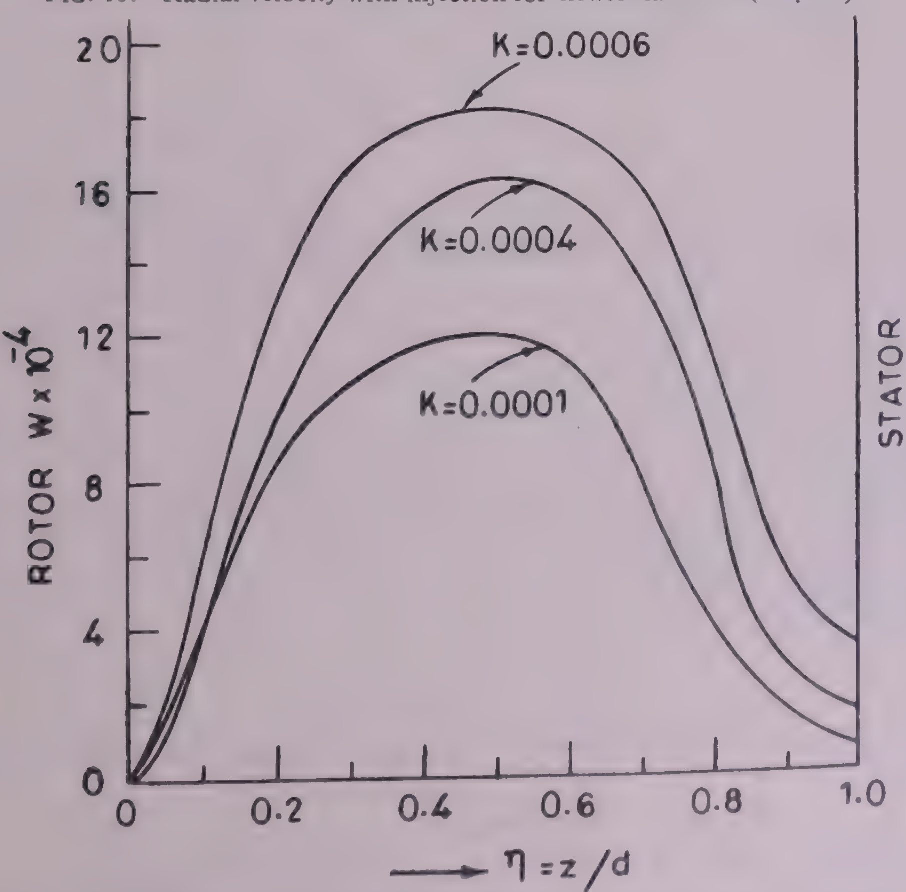


FIG. 11. Variation of axial velocity with injection for newtonian fluid ($\alpha=\beta=0$).

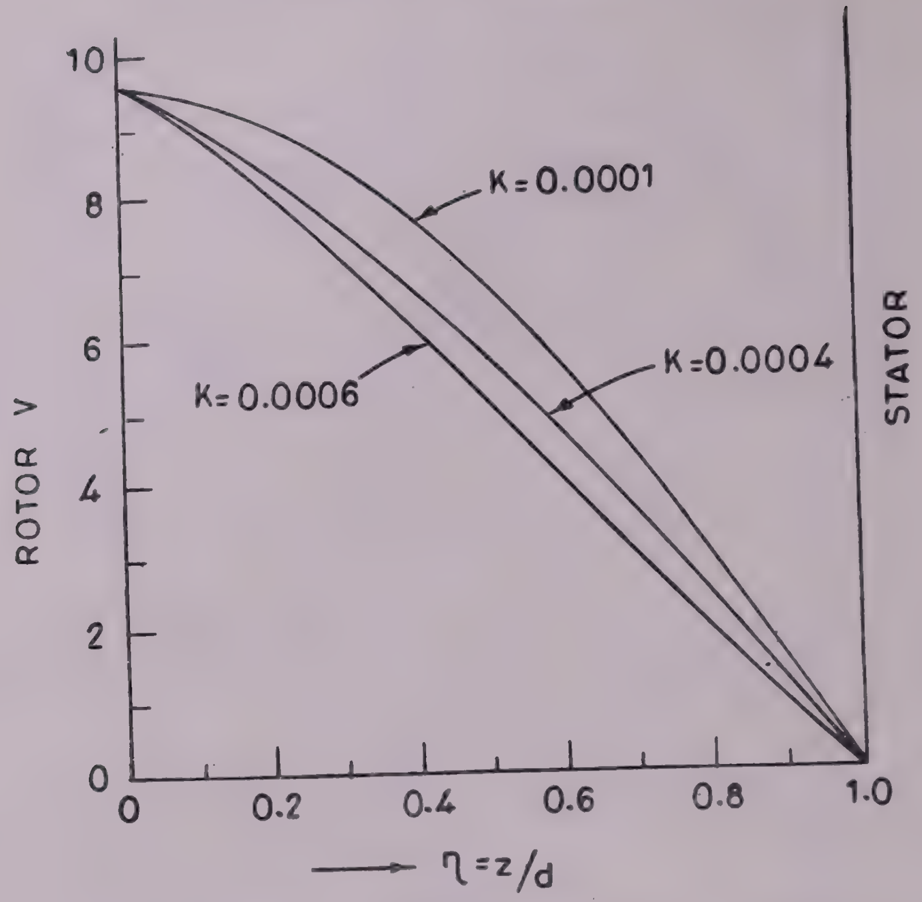


FIG. 12. Variation of transverse velocity with injection for Newtonian fluid ($\alpha=\beta=0$).

From the Fig. 10, we see that the effect of injection parameter ($k$) is to increase the radial velocity component even if we take Newtonian fluid ($\alpha = \beta = 0$). But this increase in the radial velocity component is somewhat greater as compared to that in second order incompressible fluids. But the axial velocity also increases tremendously near the rotor with an increase in injection parameter. On the other hand, it decreases with the same speed with an increase in the injection parameter near the stator. Finally, the transverse component of velocity decreases with an increase in injection parameter. But this decrease is somewhat slower as compared to that in the case of second order fluid. Moreover, the numerical value of the transverse component of velocity in Newtonian fluid is somewhat lower as compared to that in the case of second order incompressible fluids.

## ACKNOWLEDGEMENT

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# PROPAGATION OF ALFVÉN WAVES IN A REAL MAGNETOHYDRODYNAMIC FLUID

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The system of Alfvén waves in a real MHD fluid is discussed in detail. The equation of state and thermodynamic quantities of the system have been obtained with the help of the formulae of Bose-Einstein statistics and electrodynamics. It has been found that if we ignore the viscous forces from the equation of state of Alfvén waves in a real MHD fluid, the equation reduces to the equation of state of the system of Alfvén waves in an ideal MHD fluid.

## 1. INTRODUCTION

In this paper we consider Alfvén waves as a system. The energy of this system is quantized by the boundary conditions imposed on the walls of the box containing the real fluid. We calculate the density of states and then the free energy of the system with the help of the formulae of electrodynamics. This leads us to find the equation of state of the system.

## 2. DEFINITION

We consider the system of Alfvén waves in a real MHD fluid and assume that quantum of energy of an Alfvén wave exists having the value $\hbar w$ ($\hbar$ being Dirac $h$ and $w$ the angular frequency). This quantum is named as Alfvén on[1].

## 3. FORMULATION OF THE PROBLEM

We take the system of Alfvén waves in a cubic box having each side $a$ and volume $V$ and assume that the system is in thermal equilibrium with the box having perfectly reflecting and ideally conducting walls. The wave function vanishes on all sides of the box. Like photons, Alfvénons are subject to quantum statistics and especially to Bose-Einstein statistics. Moreover, they disobey pauli exclusion principle. We shall find different thermodynamic quantities of the system with the help of the formulae of electromagnetic waves.

## 4. SOLUTION OF THE PROBLEM

We write the basic equations of magnetohydrodynamics for a real MHD fluid as follows[2] :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \ldots(1)$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \frac{1}{4\pi}\{(\nabla \times \vec{B}) \times \vec{B}\} + (\eta + 1/3\,\mu)\,\nabla(\nabla \cdot \vec{v}) + \mu\nabla^2 \vec{v} \qquad \ldots(2)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \nu m\,\nabla^2 \vec{B} \qquad \ldots(3)$$

$$\nabla \cdot \vec{B} = 0 \qquad \ldots(4)$$

$$\rho T \frac{ds}{dt} = Q \qquad \ldots(5)$$

$$p = p(\rho, T) \qquad \ldots(6)$$

where $\rho = \rho(\vec{r}, t)$ is the mass density of fluid, $\vec{v} = \vec{v}(\vec{r}, t)$ its velocity, $p$ the pressure, $\vec{r}$ the displacement of the fluid, $\vec{B}$ the magnetic field, $\mu$ and $\eta$ are the first and second coefficients of viscosity, $\nu m$ the magnetic viscosity, $T$ the absolute temperature, $S$ the entropy and $Q$ the energy density dissipated per unit time. Moreover, the last two terms in eqn. (2) represent the viscous forces. We assume that the waves propagating in the fluid are the plane harmonic waves having small amplitudes. With the help of this assumption the eqns. (1)-(5) can be linearized. Hence, we write

$$\vec{v} = \vec{v}_0 + \vec{v}_1 = \vec{v}_1$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1$$

$$p = p_0 + p_1 \qquad \ldots(7)$$

$$\rho = \rho_0 + \rho_1$$

where $\vec{v}_0$, $\vec{B}_0$, $p_0$ and $\rho_0$ are constants and correspond to the uniform state of the fluid. We take $\vec{v}_0 = 0$ at equilibrium state. Moreover, $\vec{v}_1$, $\vec{B}_1$, $p_1$ and $\rho_1$ are small perturbations in the quantities $\vec{v}_0$, $\vec{B}_0$, $p_0$ and $\rho_0$ respectively. The values of these perturbations and their derivatives always remain less than the constant quantities and therefore, we neglect all but the linear terms in $\vec{v}_1$, $\vec{B}_1$, $p_1$ and $\rho_1$. Hence, eqns (1)-(5) are linearized as :

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot \vec{v}_1) = 0 \qquad \ldots(8)$$

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}\{(\nabla \times \vec{B}_1) \times \vec{B}_0\} + (\eta + 1/3\,\mu)\,\nabla(\nabla \cdot \vec{v}_1)$$

$$+ \mu \nabla^2 \vec{v}_1 \qquad ...(9)$$

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0) + \nu m \nabla^2 \vec{B}_1 \qquad ...(10)$$

$$\nabla \cdot \vec{B}_1 = 0 \qquad ...(11)$$

and

$$\rho_0 T \frac{ds}{dt} = Q. \qquad ...(12)$$

This system of homogeneous, linear partial differential equations governs the behaviour of the perturbation in space and time. The assumption that the waves propagating in the fluid are plane harmonic waves, helps us to simplify eqns. (8)–(11). We introduce the plane wave solution :

$$A = A_1 e^{i(\vec{r} \cdot \vec{k} - wt)} \qquad ...(13)$$

where $A$ is any fluctuating quantity, $A_1$ its amplitude, $w$ the angular frequency, $i = \sqrt{-1}$ and $\vec{K}$ the wave vector. Thus the simplified form of the eqns. (8)–(11) will be :

$$\rho_1 w - \rho_0 (\vec{K} \cdot \vec{v}_1) = 0 \qquad ...(14)$$

$$\rho_0 w \vec{v}_1 = p_1 \vec{K} + \frac{1}{4\pi} \{(\vec{B}_0 \cdot \vec{B}_1) \vec{K} - (\vec{K} \cdot \vec{B}_0) \vec{B}_1\} - i \mu K^2 \vec{v}_1$$

$$- i (\eta + 1/3 \mu) (\vec{K} \cdot \vec{v}_1) \vec{K} \qquad ...(15)$$

$$w \vec{B}_1 = (\vec{K} \cdot \vec{v}_1) \vec{B}_0 - (\vec{K} \cdot \vec{B}_0) \vec{v}_1 - i \nu m K^2 \vec{B}_1 \qquad ...(16)$$

$$\vec{K} \cdot \vec{B}_1 = 0. \qquad ...(17)$$

The expression for the speed of sound gives

$$p_1 = \frac{\rho_1}{\gamma} C_s^2 \qquad ...(18)$$

where $\gamma = Cp/Cv$; $Cp$ and $Cv$ are specific heats at constant pressure and volume respectively. With the help of eqns. (14)–(18), we may obtain the following relation between the kinetic and magnetic energies for all types of modes.

$$1/2 \, \rho_0 v_1^2 = \frac{\left\{C_s^2 / \gamma - i (\eta_1 + 1/3 \nu) w\right\} \rho_1^2 w/\rho_0}{2 (w + i \nu K^2)}$$

$$+ \frac{(w + i \nu m K^2)}{8\pi (w + i \nu K^2)} B_1^2 \qquad ...(19)$$

where $\eta_1 = \eta/\rho_0$ and $\nu = \mu/\rho_0$ is the kinematic viscosity. Also substitution of $\vec{B}_1$ from eqn. (16) into eqn. (15) and then simplification with the help of the eqns. (14) and (18) gives the dispersion relation for general modes, given below :

$$[\rho_0 w + i \mu K^2 - \frac{(\vec{K} \cdot \vec{B}_0)^2}{4\pi (w + i \nu m K^2)}] \vec{v}_1 = \Bigg[ \Bigg\{ \frac{\rho_0 C_s^2}{\gamma w} + \frac{B_0^2}{4\pi (w + i \nu m K^2)} - i (\eta + 1/3 \mu) \Bigg\} \vec{K} - \frac{(\vec{K} \cdot \vec{B}_0) \vec{B}_0}{4\pi (w + i \nu m K^2} \Bigg] (\vec{K} \cdot \vec{v}_1) - \frac{(\vec{K} \cdot \vec{B}_0)(\vec{B}_0 \cdot \vec{v}_1)}{4\pi (w + i \nu m K^2)} \vec{K}. \qquad ...(20)$$

Now corresponding to different modes of oscillatons this equation gives us different dispersion relations. Moreover, calculating $\vec{B}_1$ from eqn. (10) and substituting into eqn. (9) and then simplifying we can show that the velocity of Alfvén waves will be :

$$\vec{C}_A = \frac{\vec{B}_0}{\sqrt{4\pi \rho_0}}. \qquad ...(21)$$

The condition that the fluid velocity $\vec{v}_1$ must be perpendicular to the unperturbed magnetic field $\vec{B}_0$ and the wave vector $\vec{K}$ for the propagation of Alfvén waves gives :

$$\vec{K} \cdot \vec{v}_1 = 0 = \vec{B}_0 \cdot \vec{v}_1. \qquad ...(22)$$

We take $\vec{B}_0$ along $z$-axis and $\lambda$ as the angle between $\vec{B}_0$ and $\vec{K}$.

Thus,

$$\vec{K} \cdot \vec{B} = K B_0 \cos \lambda. \qquad ...(23)$$

Thus eqns. (14), (18) and (22) jointly give

$$\rho_1 = 0 = p_1. \qquad ...(24)$$

We can simplify the eqns. (19)–(20) with the help of the eqns. (23)–(24). Thus we get

$$1/2\, \rho_0 v_1^2 = \frac{(w + i \nu m K^2) B_1^2}{8\pi (w + i \nu K^2)} \qquad ...(25)$$

and

$$\nu m\, \nu K^4 + \Big[ C_A^2 \cos^2 \lambda\; i (\nu m + \nu) w \Big] K^2 - w^2 = 0. \qquad ...(26)$$

Equation (25) is the energy equation and (26) the dispersion relation for the Alfvén waves propagating in a real MHD fluid. Thus the compressibility of the fluid

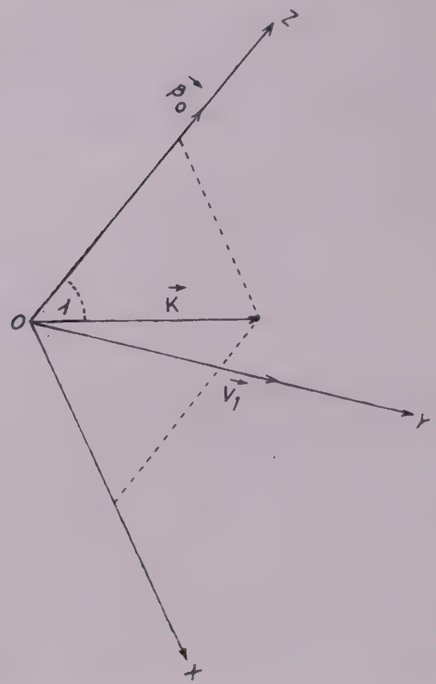

FIG. 1. Direction of the magnetic field $\mathbf{B}_0$, the wave vector $\mathbf{K}$ and the fluid velocity $\mathbf{v}_1$.

does not play any role in Alfvén waves, density and pressure perturbations do not accompany these waves, energy flow is always along the magnetic lines of forces, geometrical spreading of the energy does not take place and the medium remains unchanged thermodynamically. Moreover, there is equipartition between hydrodynamic and magnetic energies. Equation (26) gives the following relations for phase and group velocities respectively

$$u = w/K = \left\{ C_A^2 \cos^2 \lambda - i\,(\nu m + \nu)\,w \right\}^{1/2} \qquad \text{...(27)}$$

and

$$\frac{\partial w}{\partial \vec{\mathbf{K}}} = \frac{2\,C_A^2 \cos^2 \lambda - i\,(\nu m + \nu)\,w}{2\,C_A^2 \cos^2 \lambda + i\,(\nu m + \nu)\,K\,C_A \cos \lambda}\,\vec{\mathbf{C}}_A. \qquad \text{...(28)}$$

Due to the boundary conditions imposed by us on the walls of the box, the oblique propagation of MHD waves is impossible The wave propagation will be either along or across the magnetic field. Here we shall consider only the propagation of Alfvén waves along the magnetic field. Thus $\lambda = 0$ and eqn (27) can be reduced to

$$K = \frac{w}{C_A} + i\,\frac{Lw^2}{2\,C_A^3}$$

where $L = \nu m + \nu$

$$dK = \left[ \frac{1}{C_A} + i \frac{Lw}{C_A^3} \right] dw. \quad \text{...(29)}$$

Since the propagation of Alfvén waves is along one side of the cubic box only; therefore, one dimensional formula for density of states will be[3]

$$dg\,(K) = \frac{a}{\pi}\,dK.$$

Considering all types of polarizations of Alfvén waves, this will become :

$$dg\,(K) = \frac{2\,V^{1/3}}{\pi}\,dK \quad \text{...(30)}$$

where $a = V^{1/3}$.

Substitution of $dK$ from eqn. (29) into eqn. (30) gives

$$dg\,(w) = \frac{2\,V^{1/3}}{\pi} \left[ \frac{1}{C_A} + i \frac{Lw}{C_A^3} \right] dw. \quad \text{...(31)}$$

Now we calculate the thermodynamic quantities of the system. Using the formula for free energy $\xi$ of the system as given by Guggenheim[4] :

$$\xi = \theta \sum_k g_k \ln\,(1 - \exp - E_k/\theta).$$

It can be written in integral form as

$$\xi = \theta \int_0^\infty \ln\,(1 - \exp\,(-\hbar w/\theta))\,dg \quad \text{...(32)}$$

where $E_k = \hbar w$ is the energy of $k$th Alfvénon and $\theta = kT$ ($k$ being Boltzmann's constant and $T$ the absolute temperature). Substituting equation (31) into equation (32), we get

$$\xi = \frac{2V^{1/3}\,\theta}{\pi\,C_A} \int_0^\infty \ln\,(1 - \exp\,(-\hbar w/\theta))\,dw$$

$$+ \frac{2iLV^{1/3}\,\theta}{\pi\,C_A^3} \int_0^\infty w \cdot \ln\,(i - \exp\,(\hbar w/\theta))\,dw.$$

Let

$$\hbar w/\theta = x$$

$$\therefore \xi = \frac{2\, V^{1/3}\, \theta^2}{\pi\, C_A\, \hbar} \int_0^\infty \ln(1 - e^{-x})\, dx$$

$$+ \frac{2\, iLV^{1/3}\, \theta^3}{\pi\, \hbar^2\, C_A^3} \int_0^\infty x \cdot \ln(1 - e^{-x})\, dx. \qquad \ldots(33)$$

The formula for power series of logarithmic functions is given below :

$$\ln(1 - e^{-x}) = - \sum_{n=1}^{\infty} \frac{e^{-xn}}{n}.$$

We simplify the expression (33) as follows :

$$\xi = \frac{-2\, V^{1/3}\, \theta^2}{\pi\, \hbar\, C_A} \int_0^\infty \sum_{n=1}^{\infty} \frac{e^{-nx}}{n}\, dx - \frac{2\, iLV^{1/3}\, \theta^3}{\pi\, \hbar^2\, C_A^3} \int_0^\infty \sum_{n=1}^{\infty} \frac{x \cdot e^{-nx}}{n}\, dx.$$

Let

$$nx = y$$

$$\therefore \xi = \frac{-2V^{1/3}\, \theta^2}{\pi\, \hbar\, C_A} \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \int_0^\infty e^{-y}\, dy \right)$$

$$- \frac{2\, iLV^{1/3}\, \theta^3}{\pi\, \hbar^2\, C_A^3} \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \int_0^\infty ye^{-y}\, dy \right). \qquad \ldots(34)$$

The integrals in eqn. (34) can be calculated as :

$$\int_0^\infty e^{-y}\, dy = 1 = \int_0^\infty y\, e^{-y}\, dy$$

$$\therefore \xi = \frac{2\, V^{1/3}\, \theta^2}{\pi\, \hbar\, C_A} \sum_{n=1}^{\infty} \frac{1}{n^2} - i \frac{2\, LV^{1/3}\, \theta^3}{\pi\, \hbar^2\, C_A^3} \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$\ldots(35)$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202$$

$$\therefore \xi = \frac{-\pi V^{1/3} \theta^2}{3\hbar C_A} - i \frac{(2.4) LV^{1/3} \theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(36)}$$

The entropy of the system will be

$$S = -\left(\frac{\partial \xi}{\partial \theta}\right)_V$$

$$S = \frac{2\pi V^{1/3} \theta}{3 \hbar C_A} + i \frac{(7.2) LV^{1/3} \theta^2}{\pi \hbar^2 C_A^3}. \qquad \text{...(37)}$$

The pressure $p$ will be

$$p = -\left(\frac{\partial \xi}{\partial V}\right)\theta$$

$$p = \frac{\pi V^{-2/3} \theta^2}{9 \hbar C_A} + i \frac{(0.8) LV^{-2/3} \theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(38)}$$

$$pV^{2/3} = \frac{\pi \theta^2}{9 \hbar C_A} + i \frac{(0.8) L\theta^3}{\pi \hbar^2 C_A^3} \qquad \text{...(39)}$$

It is the equation of state for Alfvén waves propagating in a real MHD fluid. Thermodynamic potential $M = \xi + pV$

$$M = -\frac{2\pi V^{1/3} \theta^2}{9 \hbar C_A} - i \frac{(1.6) LV^{1/3} \theta^3}{\pi \hbar^2 C_A^3}. \qquad \text{...(40)}$$

The mean energy $\bar{E} = \xi + \theta S$

$$\therefore \bar{E} = \frac{\pi V^{1/3} \theta^2}{\pi \hbar C_A} + i \frac{(4.8) LV^{1/3} \theta^3}{\pi \hbar^2 C_A^3}. \qquad \text{...(41)}$$

The enthalpy $y = \bar{E} + pV$.

So

$$Y = \frac{4\,\pi\, V^{1/3}\,\theta^2}{9\,\hbar\, C_A} + i\,\frac{(5.6)\,LV^{1/3}\,\theta^3}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(42)$$

The specific heat at constant volume $C_V = \left[\frac{\partial \bar{E}}{\partial \theta}\right]_V$

$$C_V = \frac{2\pi\, V^{1/3}\,\theta}{3\hbar\, C_A} + i\,\frac{(14.4)\,LV^{1/3}\,\theta^2}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(43)$$

The specific heat at constant pressure $C_p$ is given by

$$C_p - C_V = -\frac{\theta\left[\frac{\partial p}{\partial \theta}\right]_V^2}{\left[\frac{\partial p}{\partial V}\right]_\theta}\,.$$

Therefore,

$$C_p = \frac{-\pi\, V^{1/3}\,\theta}{3\,\hbar^2\,C_A} + i\,\frac{(21.6)\,LV^{1/3}\,\theta^2}{\pi\,\hbar^2\,C_A^3}\,. \qquad \ldots(44)$$

The ratio of specific heats $\gamma = C_p/C_V$ can be calculated as :

$$\gamma = 1.54.$$

## 5. Conclusion

From the equation of state of the system of Alfvén waves in a real MHD fluid, we may conclude that the equation consists of real and imaginary parts. The real parts are non-viscous while the imaginary part appearing on the right hand side of the equation is due to viscous forces. If we ignore the viscous forces, eqn. (39) will be reduced to the equation of state of the system of Alfvén waves in an ideal MHD fluid[1].

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